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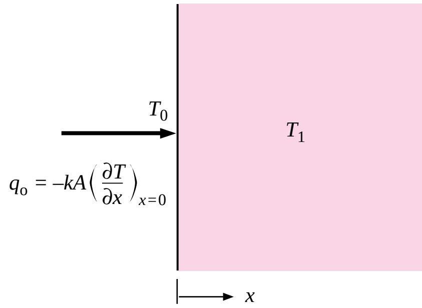
Figure 4-3 | Nomenclature for transient heat flow in a semi-infinite solid.



seek an expression for the temperature distribution in the solid as a function of time. This temperature distribution may subsequently be used to calculate heat flow at any x position in the solid as a function of time. For constant properties, the differential equation for the temperature distribution $T(x, \tau)$ is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad \textbf{[4-7]}$$

The boundary and initial conditions are

$$T(x, 0) = T_i$$
$$T(0, \tau) = T_0 \qquad \text{for } \tau > 0$$

This is a problem that may be solved by the Laplace-transform technique. The solution is given in Reference 1 as

$$\frac{T(x,\tau) - T_0}{T_i - T_0} = \operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} \qquad \textbf{[4-8]}$$

where the Gauss error function is defined as

$$\operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} = \frac{2}{\sqrt{\pi}} \int^{x/2\sqrt{\alpha\tau}} e^{-\eta^2}\, d\eta \qquad \textbf{[4-9]}$$

It will be noted that in this definition η is a dummy variable and the integral is a function of its upper limit. When the definition of the error function is inserted in Equation (4-8), the expression for the temperature distribution becomes

$$\frac{T(x,\tau) - T_0}{T_i - T_0} = \frac{2}{\sqrt{\pi}} \int^{x/2\sqrt{\alpha\tau}} e^{-\eta^2}\, d\eta \qquad \textbf{[4-10]}$$

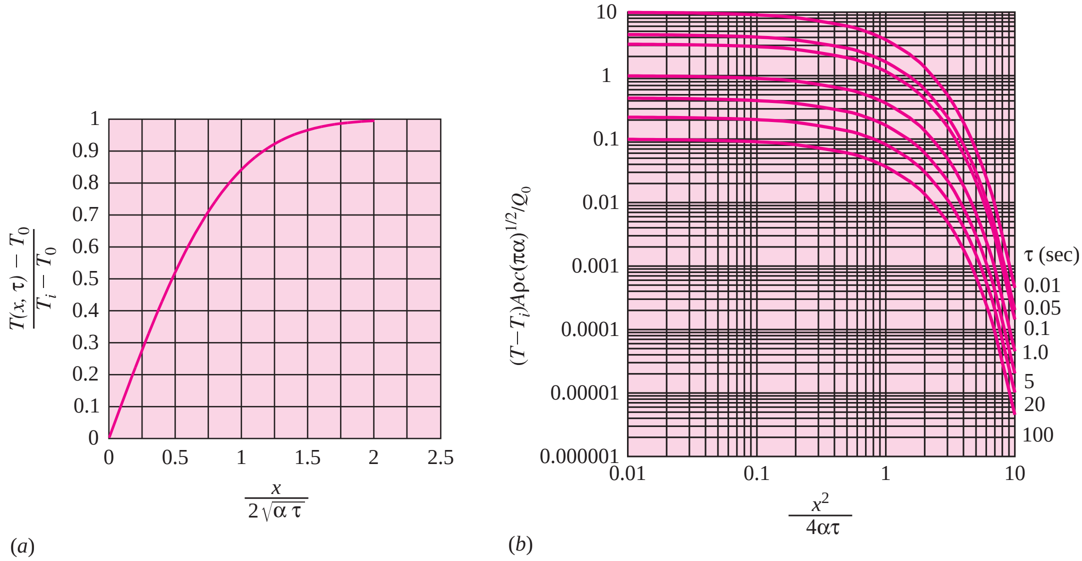
The heat flow at any x position may be obtained from

$$q_x = -kA\frac{\partial T}{\partial x}$$

Performing the partial differentiation of Equation (4-10) gives

$$\frac{\partial T}{\partial x} = (T_i - T_0)\frac{2}{\sqrt{\pi}} e^{-x^2/4\alpha\tau} \frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{\alpha\tau}}\right) \qquad \textbf{[4-11]}$$
$$= \frac{T_i - T_0}{\sqrt{\pi\alpha\tau}} e^{-x^2/4\alpha\tau}$$

Figure 4-4 | Response of semi-infinite solid to (*a*) sudden change in surface temperature and (*b*) instantaneous surface pulse of Q_0/A J/m^2.



At the surface ($x = 0$) the heat flow is

$$q_0 = \frac{kA(T_0 - T_i)}{\sqrt{\pi\alpha\tau}} \qquad \textbf{[4-12]}$$

The surface heat flux is determined by evaluating the temperature gradient at $x = 0$ from Equation (4-11). A plot of the temperature distribution for the semi-infinite solid is given in Figure 4-4. Values of the error function are tabulated in Reference 3, and an abbreviated tabulation is given in Appendix A.

Constant Heat Flux on Semi-Infinite Solid

For the same uniform initial temperature distribution, we could suddenly expose the surface to a constant surface heat flux q_0/A. The initial and boundary conditions on Equation (4-7) would then become

$$T(x,0) = T_i$$

$$\frac{q_0}{A} = -k\left.\frac{\partial T}{\partial x}\right]_{x=0} \qquad \text{for } \tau > 0$$

The solution for this case is

$$T - T_i = \frac{2q_0\sqrt{\alpha\tau/\pi}}{kA}\exp\left(\frac{-x^2}{4\alpha\tau}\right) - \frac{q_0 x}{kA}\left(1 - \text{erf}\,\frac{x}{2\sqrt{\alpha\tau}}\right) \qquad \textbf{[4-13}\boldsymbol{a}\textbf{]}$$

Energy Pulse at Surface

Equation (4-13*a*) presents the temperature response that results from a surface heat flux that remains constant with time. A related boundary condition is that of a short, instantaneous pulse of energy at the surface having a magnitude of Q_0/A. The resulting temperature response is given by

$$T - T_i = [Q_0/A\rho c(\pi\alpha\tau)^{1/2}]\exp(-x^2/4\alpha\tau) \qquad \textbf{[4-13}\boldsymbol{b}\textbf{]}$$

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In contrast to the constant-heat-flux case where the temperature increases indefinitely for all x and times, the temperature response to the instantaneous surface pulse will die out with time, or

$$T - T_i \to 0 \text{ for all } x \text{ as } \tau \to \infty$$

This rapid exponential decay behavior is illustrated in Figure 4-4*b*.

EXAMPLE 4-2 Semi-Infinite Solid with Sudden Change in Surface Conditions

A large block of steel [$k = 45$ W/m · °C, $\alpha = 1.4 \times 10^{-5}$ m^2/s] is initially at a uniform temperature of 35°C. The surface is exposed to a heat flux (*a*) by suddenly raising the surface temperature to 250°C and (*b*) through a constant surface heat flux of 3.2×10^5 W/m^2. Calculate the temperature at a depth of 2.5 cm after a time of 0.5 min for both these cases.

■ Solution

We can make use of the solutions for the semi-infinite solid given as Equations (4-8) and (4-13*a*). For case *a*,

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.025}{(2)[(1.4 \times 10^{-5})(30)]^{1/2}} = 0.61$$

The error function is determined from Appendix A as

$$\operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} = \operatorname{erf}\ 0.61 = 0.61164$$

We have $T_i = 35$°C and $T_0 = 250$°C, so the temperature at $x = 2.5$ cm is determined from Equation (4-8) as

$$\begin{aligned} T(x, \tau) &= T_0 + (T_i - T_0) \operatorname{erf} \frac{x}{2\sqrt{\alpha\tau}} \\ &= 250 + (35 - 250)(0.61164) = 118.5°\text{C} \end{aligned}$$

For the constant-heat-flux case *b*, we make use of Equation (4-13*a*). Since q_0/A is given as 3.2×10^5 W/m^2, we can insert the numerical values to give

$$\begin{aligned} T(x, \tau) &= 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} e^{-(0.61)^2} \\ &\quad - \frac{(0.025)(3.2 \times 10^5)}{45}(1 - 0.61164) \\ &= 79.3°\text{C} \qquad x = 2.5 \text{ cm}, \ \tau = 30 \text{ s} \end{aligned}$$

For the constant-heat-flux case the *surface* temperature after 30 s would be evaluated with $x = 0$ in Equation (4-13*a*). Thus,

$$T(x = 0) = 35 + \frac{(2)(3.2 \times 10^5)[(1.4 \times 10^{-5})(30)/\pi]^{1/2}}{45} = 199.4°\text{C}$$

EXAMPLE 4-3 Pulsed Energy at Surface of Semi-Infinite Solid

An instantaneous laser pulse of 10 MJ/m^2 is imposed on a slab of stainless steel having properties of $\rho = 7800$ kg/m^3, $c = 460$ J/kg · °C, and $\alpha = 0.44 \times 10^{-5}$ m^2/s. The slab is initially at a uniform temperature of 40 °C. Estimate the temperature at the surface and at a depth of 2.0 mm after a time of 2 s.

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Solution

This problem is a direct application of Equation (4-13*b*). We have $Q_0/A = 10^7$ J/m^2 and at $x = 0$

$$T_0 - T_i = Q_0/A\rho c(\pi\alpha\tau)^{1.2}$$
$$= 10^7/(7800)(460)[\pi(0.44 \times 10^{-5})(2)]^{0.5} = 530°\text{C}$$

and

$$T_0 = 40 + 530 = 570°\text{C}$$

At $x = 2.0\,\text{mm} = 0.002\,\text{m}$,

$$T - T_i = (530)\exp[-(0.002)^2/(4)(0.44 \times 10^{-5})(2)] = 473°\text{C}$$

and

$$T = 40 + 473 = 513°\text{C}$$

Heat Removal from Semi-Infinite Solid — EXAMPLE 4-4

A large slab of aluminum at a uniform temperature of 200°C suddenly has its surface temperature lowered to 70°C. What is the total heat removed from the slab per unit surface area when the temperature at a depth 4.0 cm has dropped to 120°C?

Solution

We first find the time required to attain the 120°C temperature and then integrate Equation (4-12) to find the total heat removed during this time interval. For aluminum,

$$\alpha = 8.4 \times 10^{-5}\,\text{m}^2/\text{s} \qquad k = 215\,\text{W/m}\cdot°\text{C}\,[124\,\text{Btu/h}\cdot\text{ft}\cdot°\text{F}]$$

We also have

$$T_i = 200°\text{C} \qquad T_0 = 70°\text{C} \qquad T(x, \tau) = 120°\text{C}$$

Using Equation (4-8) gives

$$\frac{120 - 70}{200 - 70} = \text{erf}\,\frac{x}{2\sqrt{\alpha\tau}} = 0.3847$$

From Figure 4-4 or Appendix A,

$$\frac{x}{2\sqrt{\alpha\tau}} = 0.3553$$

and

$$\tau = \frac{(0.04)^2}{(4)(0.3553)^2(8.4 \times 10^{-5})} = 37.72\,\text{s}$$

The total heat removed at the surface is obtained by integrating Equation (4-12):

$$\frac{Q_0}{A} = \int_0^\tau \frac{q_0}{A}\,d\tau = \int_0^\tau \frac{k(T_0 - T_i)}{\sqrt{\pi\alpha\tau}}\,d\tau = 2k(T_0 - T_i)\sqrt{\frac{\tau}{\pi\alpha}}$$
$$= (2)(215)(70 - 200)\left[\frac{37.72}{\pi(8.4 \times 10^{-5})}\right]^{1/2} = -21.13 \times 10^6\,\text{J/m}^2 \quad [-1861\,\text{Btu/ft}^2]$$

4-4 | CONVECTION BOUNDARY CONDITIONS

In most practical situations the transient heat-conduction problem is connected with a convection boundary condition at the surface of the solid. Naturally, the boundary conditions for the differential equation must be modified to take into account this convection heat

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transfer at the surface. For the semi-infinite-solid problem, the convection boundary condition would be expressed by

Heat convected into surface = heat conducted into surface

or

$$hA(T_\infty - T)_{x=0} = -kA \left.\frac{\partial T}{\partial x}\right]_{x=0} \qquad \textbf{[4-14]}$$

The solution for this problem is rather involved and is worked out in detail by Schneider [1]. The result is

$$\frac{T - T_i}{T_\infty - T_i} = 1 - \operatorname{erf} X - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha\tau}{k^2}\right)\right] \times \left[1 - \operatorname{erf}\left(X + \frac{h\sqrt{\alpha\tau}}{k}\right)\right] \qquad \textbf{[4-15]}$$

where

$X = x/(2\sqrt{\alpha\tau})$
$T_i =$ initial temperature of solid
$T_\infty =$ environment temperature

This solution is presented in graphical form in Figure 4-5.

Solutions have been worked out for other geometries. The most important cases are those dealing with (1) plates whose thickness is small in relation to the other dimensions, (2) cylinders where the diameter is small compared to the length, and (3) spheres. Results of analyses for these geometries have been presented in graphical form by Heisler [2], and nomenclature for the three cases is illustrated in Figure 4-6. In all cases the convection environment temperature is designated as T_∞ and the center temperature for $x = 0$ or $r = 0$ is T_0. At time zero, each solid is assumed to have a uniform initial temperature T_i. Temperatures in the solids are given in Figures 4-7 to 4-13 as functions of time and spatial position. In these charts we note the definitions

$$\theta = T(x, \tau) - T_\infty \qquad \text{or} \qquad T(r, \tau) - T_\infty$$
$$\theta_i = T_i - T_\infty$$
$$\theta_0 = T_0 - T_\infty$$

If a centerline temperature is desired, only one chart is required to obtain a value for θ_0 and then T_0. To determine an off-center temperature, two charts are required to calculate the product

$$\frac{\theta}{\theta_i} = \frac{\theta_0}{\theta_i}\frac{\theta}{\theta_0}$$

For example, Figures 4-7 and 4-10 would be employed to calculate an off-center temperature for an infinite plate.

The heat losses for the infinite plate, infinite cylinder, and sphere are given in Figures 4-14 to 4-16, where Q_0 represents the initial internal energy content of the body in reference to the environment temperature

$$Q_0 = \rho c V(T_i - T_\infty) = \rho c V\theta_i \qquad \textbf{[4-16]}$$

In these figures Q is the actual heat lost by the body in time τ.

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Figure 4-5 | Temperature distribution in the semi-infinite solid with convection boundary condition.

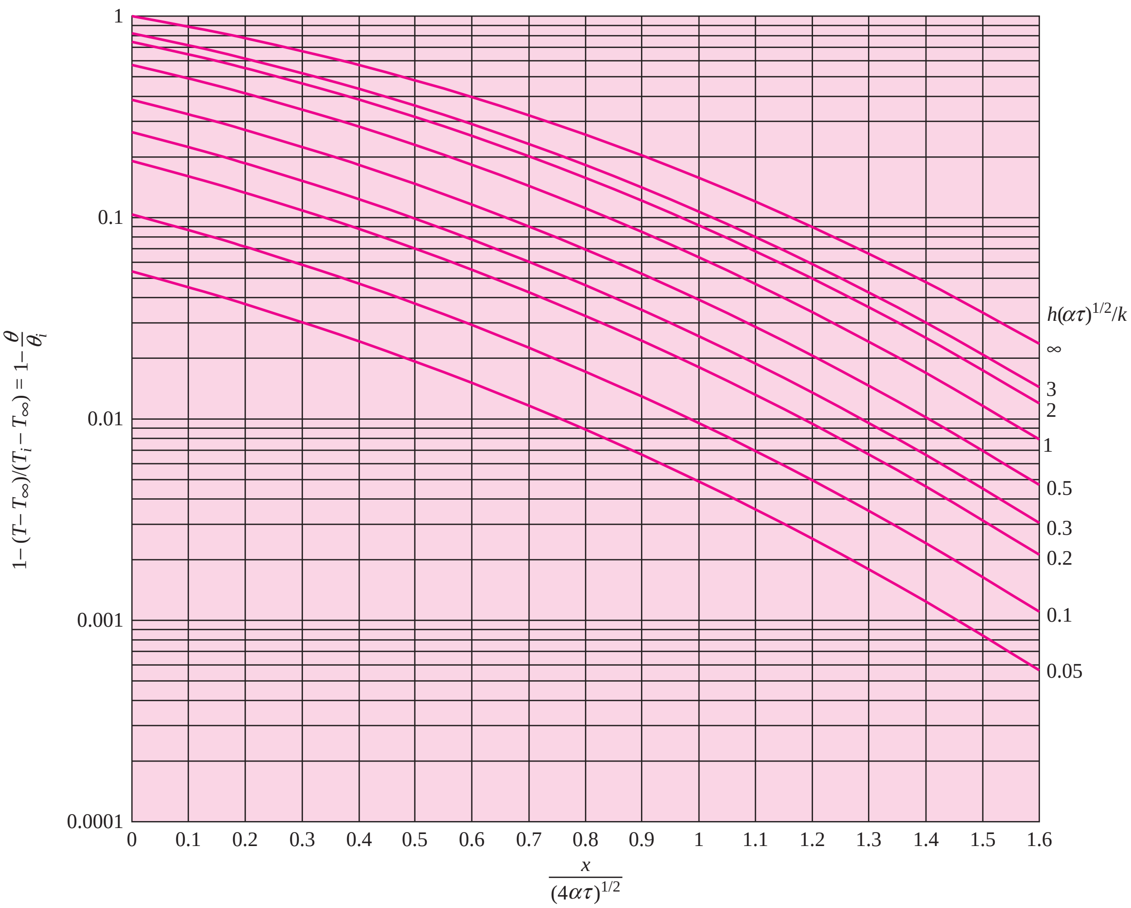

Figure 4-6 | Nomenclature for one-dimensional solids suddenly subjected to convection environment at T_∞: (*a*) infinite plate of thickness $2L$; (*b*) infinite cylinder of radius r_0; (*c*) sphere of radius r_0.

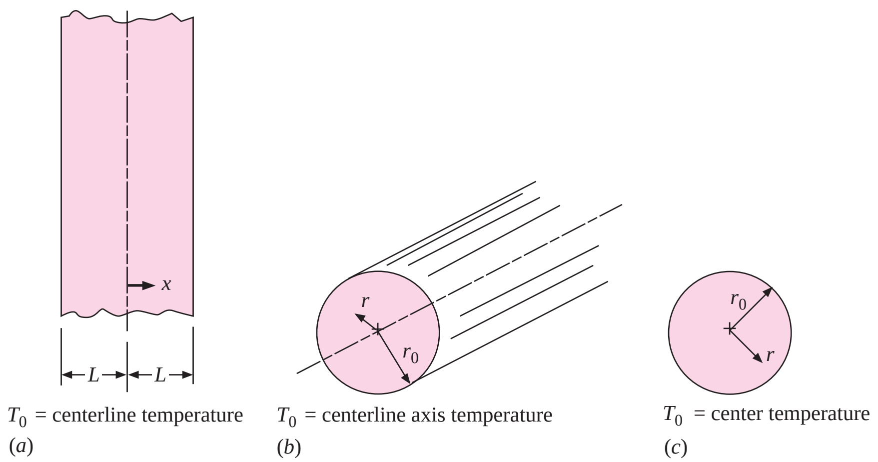

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Figure 4-7 | Midplane temperature for an infinite plate of thickness $2L$: (a) full scale.

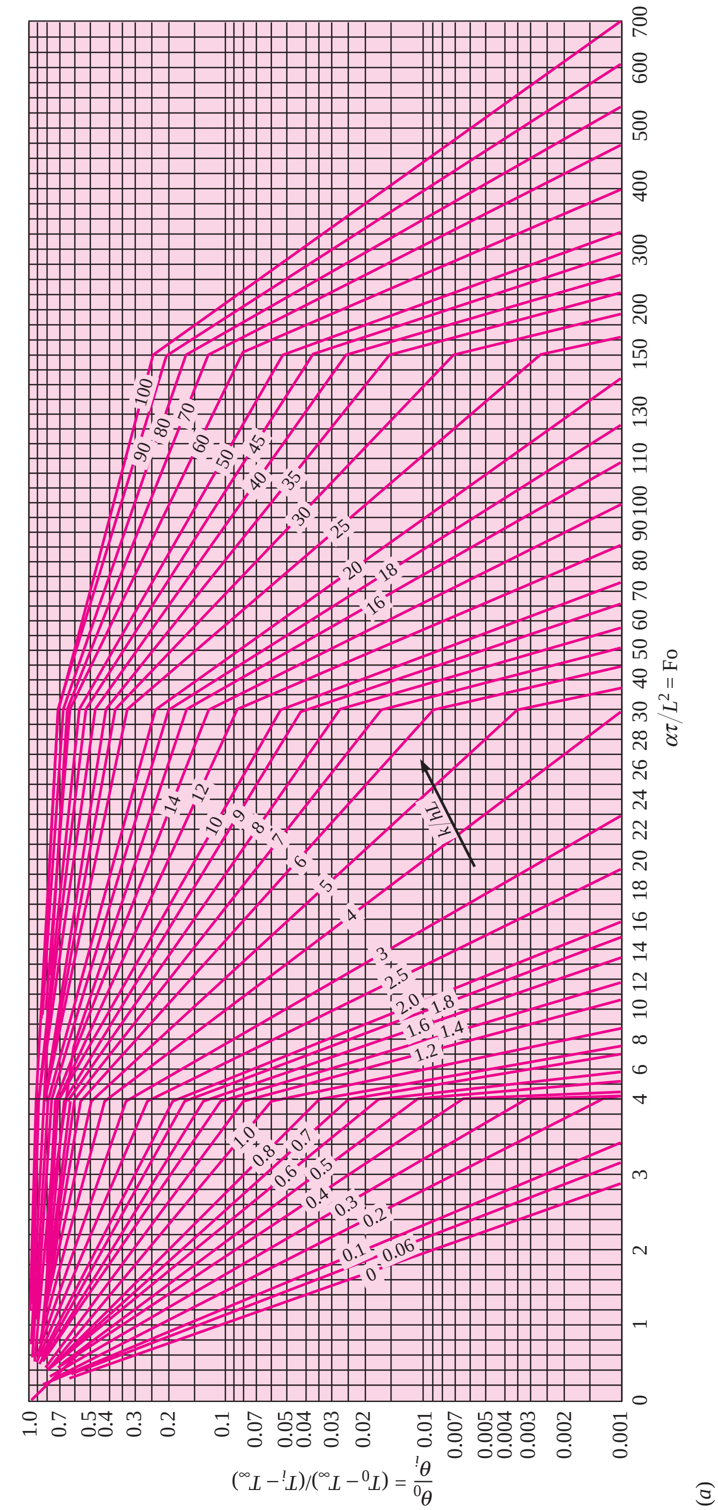

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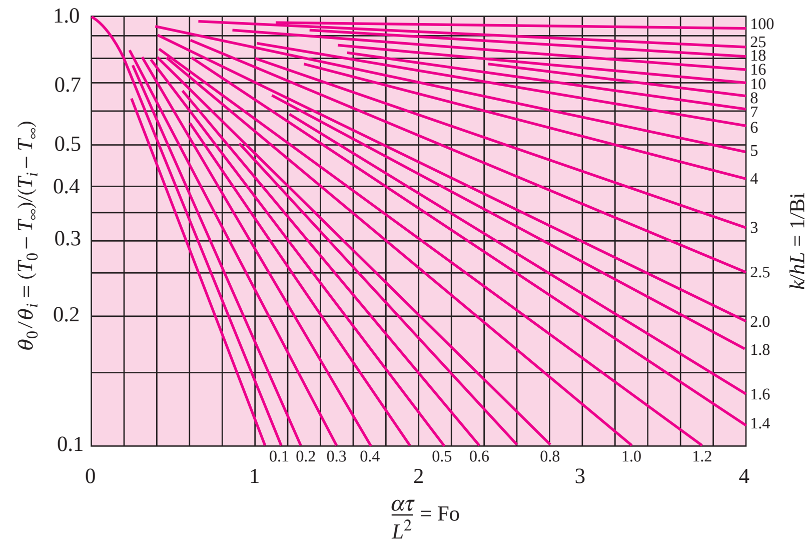
Figure 4-7 | (*Continued*). (*b*) expanded scale for 0 < Fo < 4, from Reference 2.



(*b*)

If one considers the solid as behaving as a lumped capacity during the cooling or heating process, that is, small internal resistance compared to surface resistance, the exponential cooling curve of Figure 4-5 may be replotted in expanded form, as shown in Figure 4-13 using the Biot-Fourier product as the abscissa. We note that the following parameters apply for the bodies considered in the Heisler charts.

$$(A/V)_{\text{inf plate}} = 1/L$$
$$(A/V)_{\text{inf cylinder}} = 2/r_0$$
$$(A/V)_{\text{sphere}} = 3/r_0$$

Obviously, there are many other practical heating and cooling problems of interest. The solutions for a large number of cases are presented in graphical form by Schneider [7], and readers interested in such calculations will find this reference to be of great utility.

The Biot and Fourier Numbers

A quick inspection of Figures 4-5 to 4-16 indicates that the dimensionless temperature profiles and heat flows may all be expressed in terms of two dimensionless parameters called the Biot and Fourier numbers:

$$\text{Biot number} = \text{Bi} = \frac{hs}{k}$$
$$\text{Fourier number} = \text{Fo} = \frac{\alpha\tau}{s^2} = \frac{k\tau}{\rho c s^2}$$

In these parameters s designates a characteristic dimension of the body; for the plate it is the half-thickness, whereas for the cylinder and sphere it is the radius. The Biot number compares the relative magnitudes of surface-convection and internal-conduction resistances

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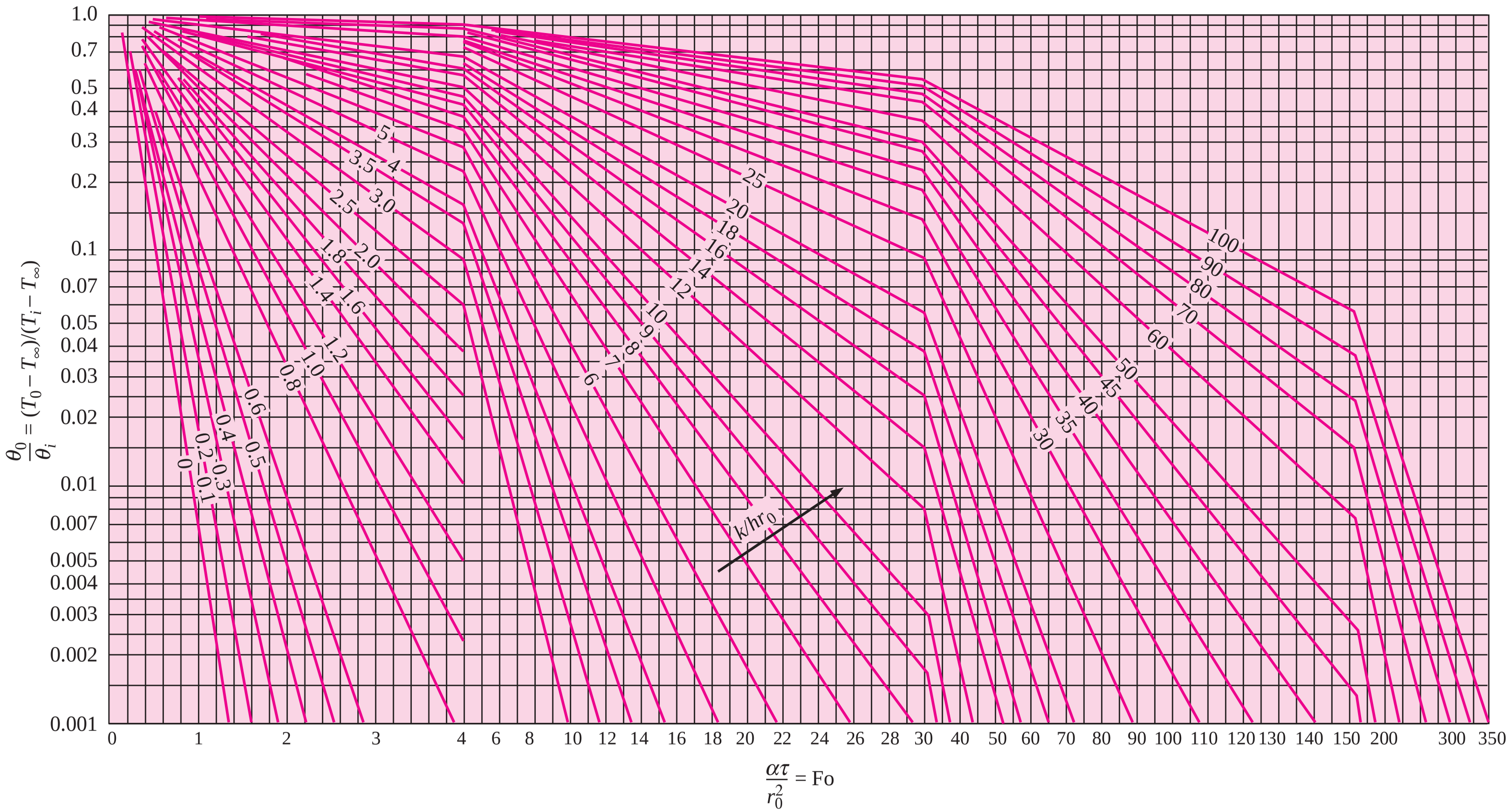
Figure 4-8 | Axis temperature for an infinite cylinder of radius r_0: (*a*) full scale.



(*a*)

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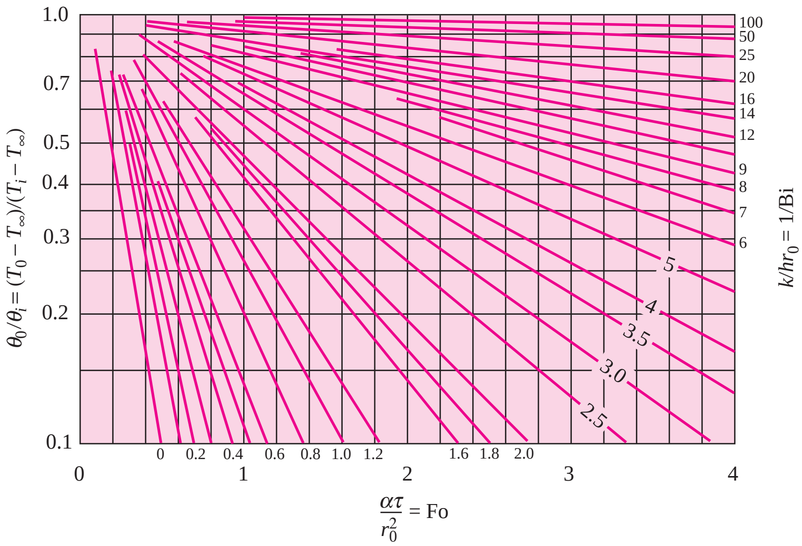
Figure 4-8 | (*Continued*). (*b*) expanded scale for 0 < Fo < 4, from Reference 2.



(*b*)

to heat transfer. The Fourier modulus compares a characteristic body dimension with an approximate temperature-wave penetration depth for a given time τ.

A very low value of the Biot modulus means that internal-conduction resistance is negligible in comparison with surface-convection resistance. This in turn implies that the temperature will be nearly uniform throughout the solid, and its behavior may be approximated by the lumped-capacity method of analysis. It is interesting to note that the exponent of Equation (4-5) may be expressed in terms of the Biot and Fourier numbers if one takes the ratio V/A as the characteristic dimension s. Then,

$$\frac{hA}{\rho c V}\tau = \frac{h\tau}{\rho c s} = \frac{hs}{k}\frac{k\tau}{\rho c s^2} = \text{Bi Fo}$$

Applicability of the Heisler Charts

The calculations for the Heisler charts were performed by truncating the infinite series solutions for the problems into a few terms. This restricts the applicability of the charts to values of the Fourier number greater than 0.2.

$$\text{Fo} = \frac{\alpha\tau}{s^2} > 0.2$$

For smaller values of this parameter the reader should consult the solutions and charts given in the references at the end of the chapter. Calculations using the truncated series solutions directly are discussed in Appendix C.

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Figure 4-9 | Center temperature for a sphere of radius r_0: (a) full scale.

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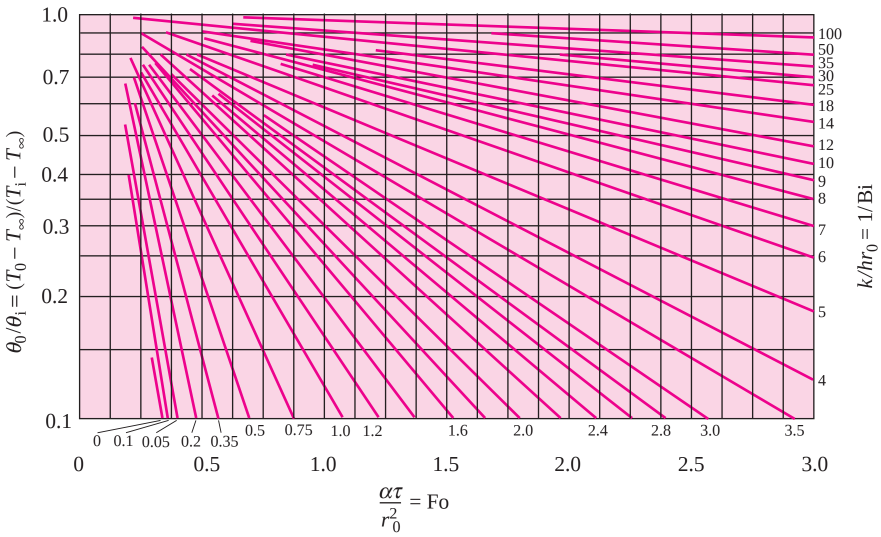
Figure 4-9 | (*Continued*). (*b*) expanded scale for 0 < Fo < 3, from Reference 2.



(*b*)

Figure 4-10 | Temperature as a function of center temperature in an infinite plate of thickness $2L$, from Reference 2.

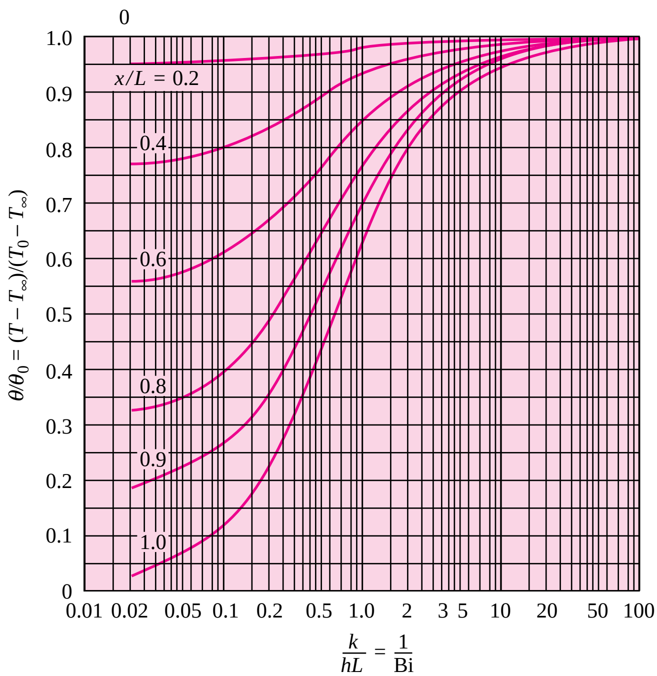


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Figure 4-11 | Temperature as a function of axis temperature in an infinite cylinder of radius r_0, from Reference 2.

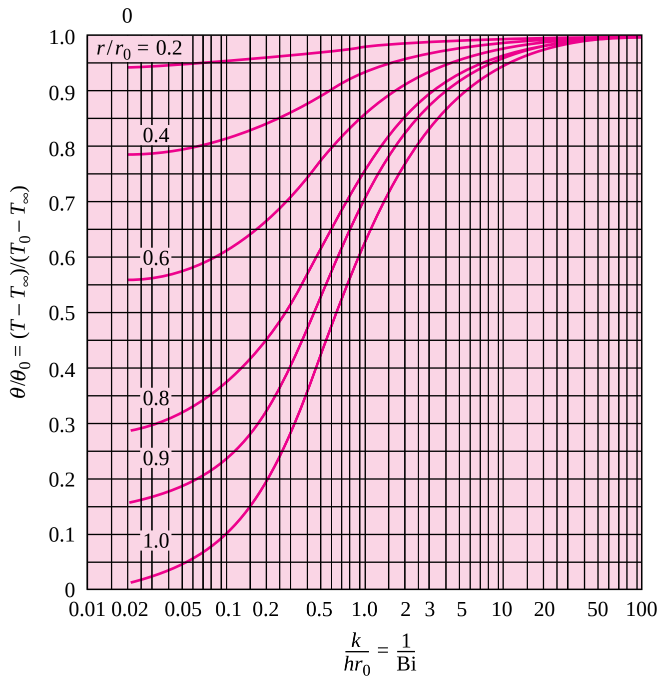

Figure 4-12 | Temperature as a function of center temperature for a sphere of radius r_0, from Reference 2.

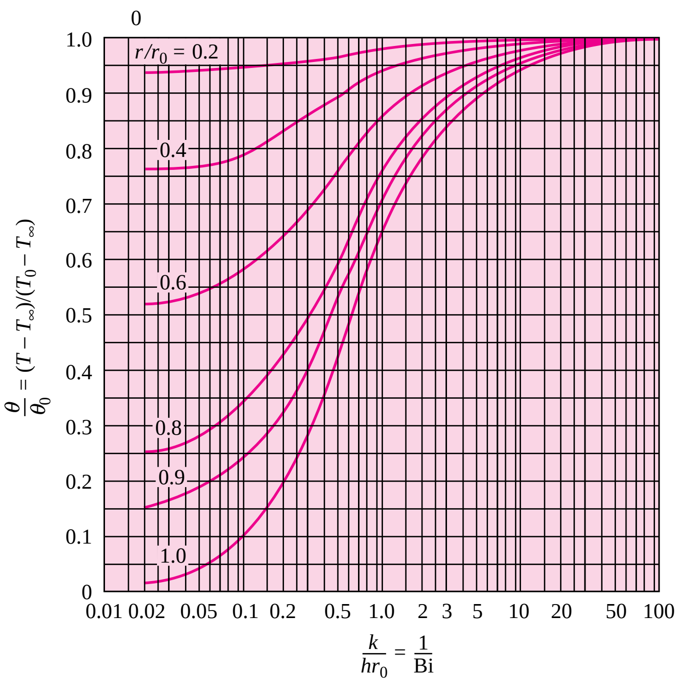

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Figure 4-13 | Temperature variation with time for solids that may be treated as lumped capacities: (*a*) $0 < \mathrm{BiFo} < 10$, (*b*) $0.1 < \mathrm{BiFo} < 1.0$, (*c*) $0 < \mathrm{BiFo} < 0.1$. Note: $(A/V)_{\text{inf plate}} = 1/L$, $(A/V)_{\text{inf cyl}} = 2/r_0$, $(A/V)_{\text{sphere}} = 3/r_0$. See Equations (4-5) and (4-6).

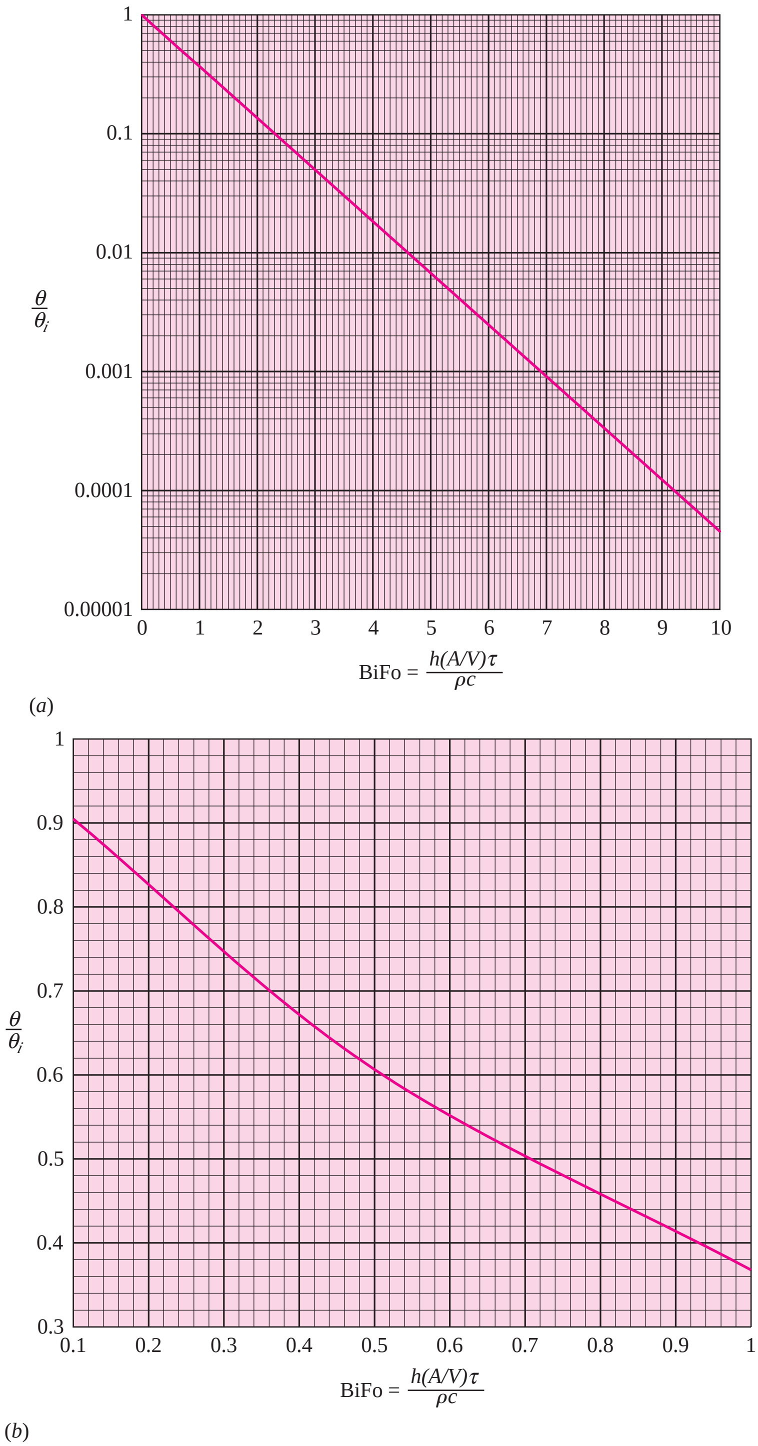

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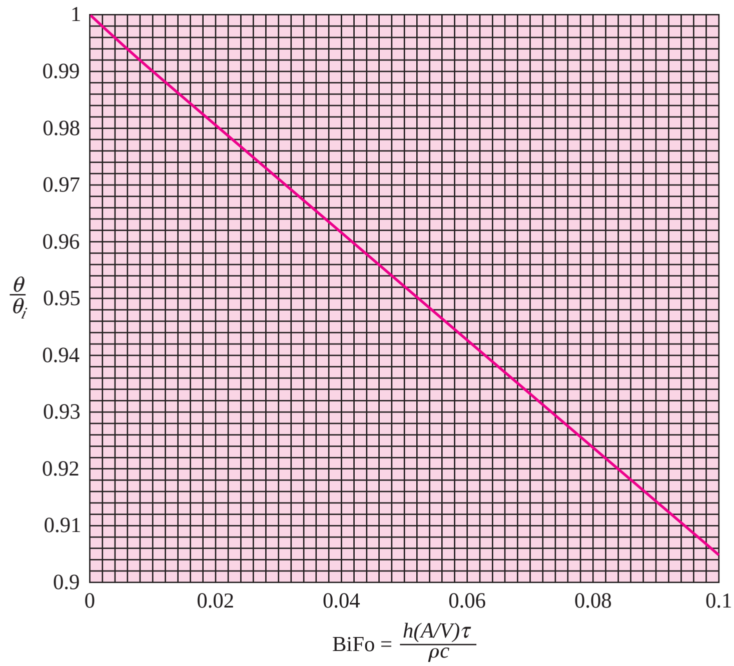
Figure 4-13 | (*Continued*).



(*c*)

Figure 4-14 | Dimensionless heat loss Q/Q_0 of an infinite plane of thickness $2L$ with time, from Reference 6.

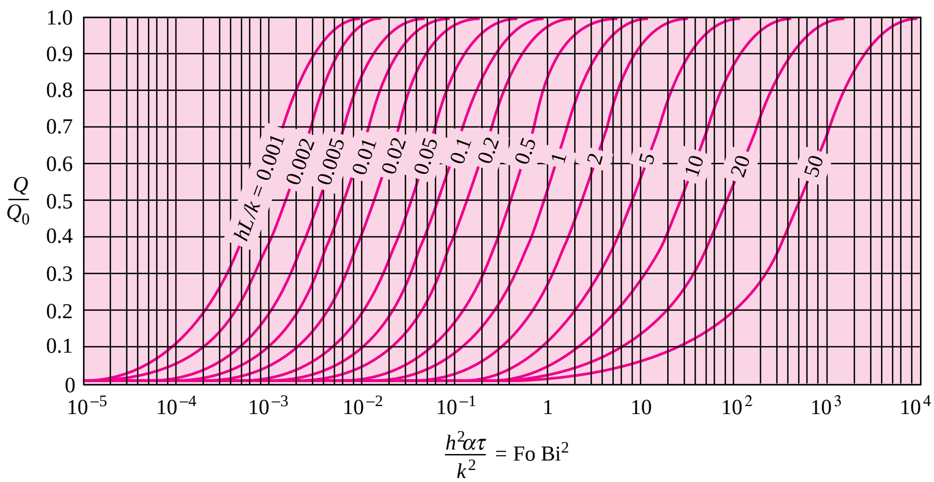


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Figure 4-15 | Dimensionlesss heat loss Q/Q_0 of an infinite cylinder of radius r_0 with time, from Reference 6.

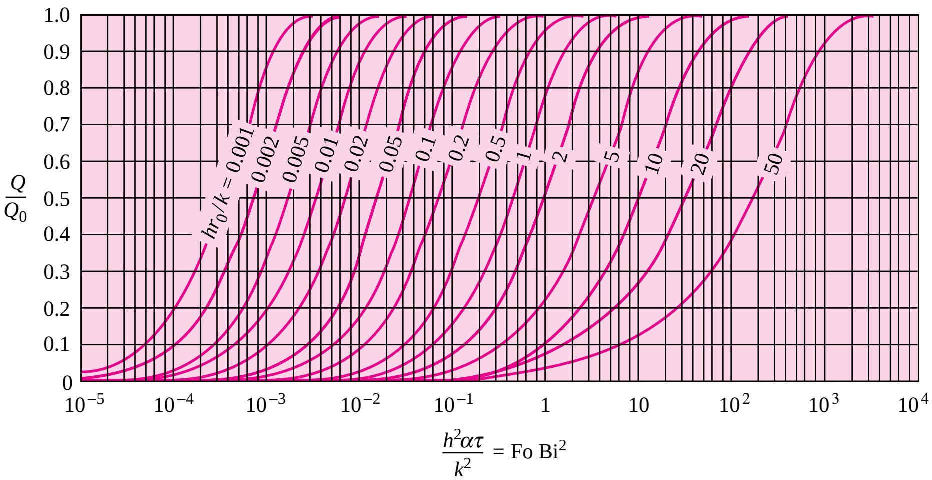
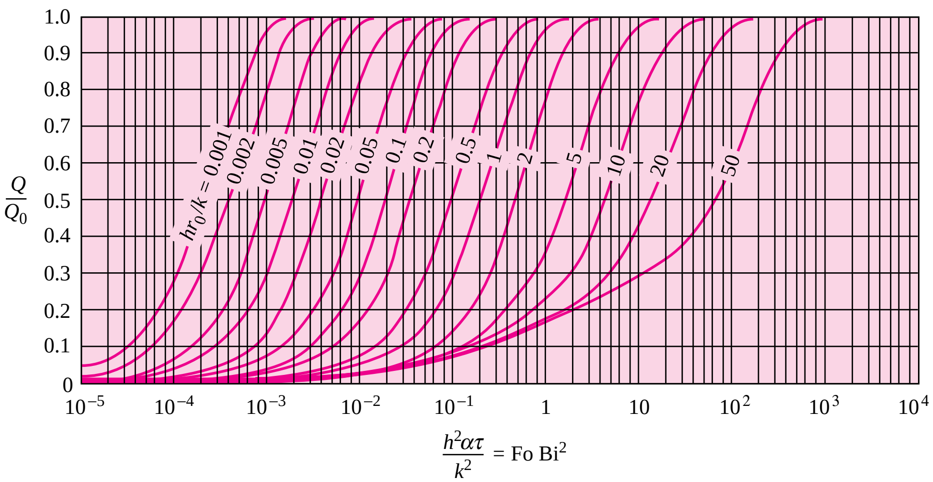


Figure 4-16 | Dimensionless heat loss Q/Q_0 of a sphere of radius r_0 with time, from Reference 6.



Sudden Exposure of Semi-Infinite Slab to Convection

EXAMPLE 4-5

The slab of Example 4-4 is suddenly exposed to a convection-surface environment of 70°C with a heat-transfer coefficient of 525 W/m^2 · °C. Calculate the time required for the temperature to reach 120°C at the depth of 4.0 cm for this circumstance.

■ Solution

We may use either Equation (4-15) or Figure 4-5 for solution of this problem, but Figure 4-5 is easier to apply because the time appears in two terms. Even when the figure is used, an iterative procedure is required because the time appears in both of the variables $h\sqrt{\alpha\tau}/k$ and $x/(2\sqrt{\alpha\tau})$.

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We seek the value of τ such that

$$\frac{T - T_i}{T_\infty - T_i} = \frac{120 - 200}{70 - 200} = 0.615 \qquad \textbf{[a]}$$

We therefore try values of τ and obtain readings of the temperature ratio from Figure 4-5 until agreement with Equation (a) is reached. The iterations are listed below. Values of k and α are obtained from Example 4-4.

τ, s	$\frac{h\sqrt{\alpha\tau}}{k}$	$\frac{x}{2\sqrt{\alpha\tau}}$	$\frac{T - T_i}{T_\infty - T_i}$ from Figure 4-5
1000	0.708	0.069	0.41
3000	1.226	0.040	0.61
4000	1.416	0.035	0.68

Consequently, the time required is approximately 3000 s.

EXAMPLE 4-6 Aluminum Plate Suddenly Exposed to Convection

A large plate of aluminum 5.0 cm thick and initially at 200°C is suddenly exposed to the convection environment of Example 4-5. Calculate the temperature at a depth of 1.25 cm from one of the faces 1 min after the plate has been exposed to the environment. How much energy has been removed per unit area from the plate in this time?

■ Solution

The Heisler charts of Figures 4-7 and 4-10 may be used for solution of this problem. We first calculate the center temperature of the plate, using Figure 4-7, and then use Figure 4-10 to calculate the temperature at the specified x position. From the conditions of the problem we have

$$\theta_i = T_i - T_\infty = 200 - 70 = 130 \qquad \alpha = 8.4 \times 10^{-5}\ \mathrm{m^2/s} \quad [3.26\ \mathrm{ft^2/h}]$$

$$2L = 5.0\ \mathrm{cm} \qquad L = 2.5\ \mathrm{cm} \qquad \tau = 1\ \mathrm{min} = 60\ \mathrm{s}$$

$$k = 215\ \mathrm{W/m \cdot {}^\circ C} \quad [124\ \ \mathrm{Btu/h \cdot ft \cdot {}^\circ F}]$$

$$h = 525\ \mathrm{W/m^2 \cdot {}^\circ C} \quad [92.5\ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

$$x = 2.5 - 1.25 = 1.25\ \mathrm{cm}$$

Then

$$\frac{\alpha\tau}{L^2} = \frac{(8.4 \times 10^{-5})(60)}{(0.025)^2} = 8.064 \qquad \frac{k}{hL} = \frac{215}{(525)(0.025)} = 16.38$$

$$\frac{x}{L} = \frac{1.25}{2.5} = 0.5$$

From Figure 4-7

$$\frac{\theta_0}{\theta_i} = 0.61$$

$$\theta_0 = T_0 - T_\infty = (0.61)(130) = 79.3$$

From Figure 4-10 at $x/L = 0.5$,

$$\frac{\theta}{\theta_0} = 0.98$$

and

$$\theta = T - T_\infty = (0.98)(79.3) = 77.7$$

$$T = 77.7 + 70 = 147.7^\circ\mathrm{C}$$

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We compute the energy lost by the slab by using Figure 4-14. For this calculation we require the following properties of aluminum:

$$\rho = 2700 \text{ kg/m}^3 \qquad c = 0.9 \text{ kJ/kg} \cdot {}^\circ\text{C}$$

For Figure 4-14 we need

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4\times10^{-5})(60)}{(215)^2} = 0.03 \qquad \frac{hL}{k} = \frac{(525)(0.025)}{215} = 0.061$$

From Figure 4-14

$$\frac{Q}{Q_0} = 0.41$$

For unit area

$$\begin{aligned}\frac{Q_0}{A} &= \frac{\rho c V\theta_i}{A} = \rho c(2L)\theta_i \\ &= (2700)(900)(0.05)(130) \\ &= 15.8\times10^6 \text{ J/m}^2\end{aligned}$$

so that the heat removed per unit surface area is

$$\frac{Q}{A} = (15.8\times10^6)(0.41) = 6.48\times10^6 \text{ J/m}^2 \quad [571 \text{ Btu/ft}^2]$$

Long Cylinder Suddenly Exposed to Convection

EXAMPLE 4-7

A long aluminum cylinder 5.0 cm in diameter and initially at 200°C is suddenly exposed to a convection environment at 70°C and $h = 525$ W/m$^2 \cdot$°C. Calculate the temperature at a radius of 1.25 cm and the heat lost per unit length 1 min after the cylinder is exposed to the environment.

■ Solution

This problem is like Example 4-6 except that Figures 4-8 and 4-11 are employed for the solution. We have

$$\theta_i = T_i - T_\infty = 200 - 70 = 130 \qquad \alpha = 8.4\times10^{-5} \text{ m}^2\text{/s}$$
$$r_0 = 2.5 \text{ cm} \qquad \tau = 1 \text{ min} = 60 \text{ s}$$
$$k = 215 \text{ W/m}\cdot{}^\circ\text{C} \qquad h = 525 \text{ W/m}^2\cdot{}^\circ\text{C} \qquad r = 1.25 \text{ cm}$$
$$\rho = 2700 \text{ kg/m}^3 \qquad c = 0.9 \text{ kJ/kg}\cdot{}^\circ\text{C}$$

We compute

$$\frac{\alpha\tau}{r_0^2} = \frac{(8.4\times10^{-5})(60)}{(0.025)^2} = 8.064 \qquad \frac{k}{hr_0} = \frac{215}{(525)(0.025)} = 16.38$$
$$\frac{r}{r_0} = \frac{1.25}{2.5} = 0.5$$

From Figure 4-8

$$\frac{\theta_0}{\theta_i} = 0.38$$

and from Figures 4-11 at $r/r_0 = 0.5$

$$\frac{\theta}{\theta_0} = 0.98$$

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so that

$$\frac{\theta}{\theta_i} = \frac{\theta_0}{\theta_i}\frac{\theta}{\theta_0} = (0.38)(0.98) = 0.372$$

and

$$\theta = T - T_\infty = (0.372)(130) = 48.4$$
$$T = 70 + 48.4 = 118.4°\text{C}$$

To compute the heat lost, we determine

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4\times10^{-5})(60)}{(215)^2} = 0.03 \qquad \frac{hr_0}{k} = \frac{(525)(0.025)}{215} = 0.061$$

Then from Figure 4-15

$$\frac{Q}{Q_0} = 0.65$$

For unit length

$$\frac{Q_0}{L} = \frac{\rho c V\theta_i}{L} = \rho c\pi r_0^2\theta_i = (2700)(900)\pi(0.025)^2(130) = 6.203\times10^5 \text{ J/m}$$

and the actual heat lost per unit length is

$$\frac{Q}{L} = (6.203\times10^5)(0.65) = 4.032\times10^5 \text{ J/m} \qquad [116.5 \text{ Btu/ft}]$$

4-5 | MULTIDIMENSIONAL SYSTEMS

The Heisler charts discussed in Section 4-4 may be used to obtain the temperature distribution in the infinite plate of thickness $2L$, in the long cylinder, or in the sphere. When a wall whose height and depth dimensions are not large compared with the thickness or a cylinder whose length is not large compared with its diameter is encountered, additional space coordinates are necessary to specify the temperature, the charts no longer apply, and we are forced to seek another method of solution. Fortunately, it is possible to combine the solutions for the one-dimensional systems in a very straightforward way to obtain solutions for the multidimensional problems.

It is clear that the infinite rectangular bar in Figure 4-17 can be formed from two infinite plates of thickness $2L_1$ and $2L_2$, respectively. The differential equation governing this situation would be

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau} \qquad \textbf{[4-17]}$$

and to use the separation-of-variables method to effect a solution, we should assume a product solution of the form

$$T(x, z, \tau) = X(x)Z(z)\Theta(\tau)$$

It can be shown that the dimensionless temperature distribution may be expressed as a product of the solutions for two plate problems of thickness $2L_1$ and $2L_2$, respectively:

$$\left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{\text{bar}} = \left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{2L_1\ plate}\left(\frac{T - T_\infty}{T_i - T_\infty}\right)_{2L_2\ plate} \qquad \textbf{[4-18]}$$

where T_i is the initial temperature of the bar and T_∞ is the environment temperature.

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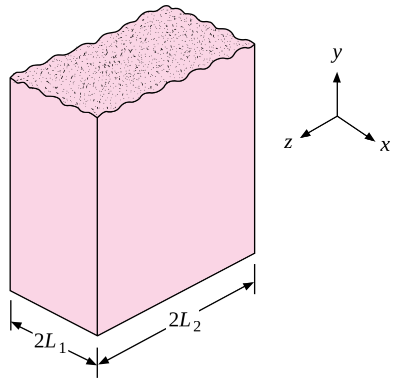
Figure 4-17 | Infinite rectangular bar.



For two infinite plates the respective differential equations would be

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_1}{\partial \tau} \qquad \frac{\partial^2 T_2}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T_2}{\partial \tau} \tag{4-19}$$

and the product solutions assumed would be

$$T_1 = T_1(x, \tau) \qquad T_2 = T_2(z, \tau) \tag{4-20}$$

We shall now show that the product solution to Equation (4-17) can be formed from a simple product of the functions (T_1, T_2), that is,

$$T(x, z, \tau) = T_1(x, \tau)T_2(z, \tau) \tag{4-21}$$

The appropriate derivatives for substitution in Equation (4-17) are obtained from Equation (4-21) as

$$\frac{\partial^2 T}{\partial x^2} = T_2\frac{\partial^2 T_1}{\partial x^2} \qquad \frac{\partial^2 T}{\partial z^2} = T_1\frac{\partial^2 T_2}{\partial z^2}$$

$$\frac{\partial T}{\partial \tau} = T_1\frac{\partial T_2}{\partial \tau} + T_2\frac{\partial T_1}{\partial \tau}$$

Using Equations (4-19), we have

$$\frac{\partial T}{\partial \tau} = \alpha T_1\frac{\partial^2 T_2}{\partial z^2} + \alpha T_2\frac{\partial^2 T_1}{\partial x^2}$$

Substituting these relations in Equation (4-17) gives

$$T_2\frac{\partial^2 T_1}{\partial x_2} + T_1\frac{\partial^2 T_2}{\partial z^2} = \frac{1}{\alpha}\left(\alpha T_1\frac{\partial^2 T_2}{\partial z^2} + \alpha T_2\frac{\partial^2 T_1}{\partial x^2}\right)$$

or the assumed product solution of Equation (4-21) does indeed satisfy the original differential equation (4-17). This means that the dimensionless temperature distribution for the infinite rectangular bar may be expressed as a product of the solutions for two plate problems of thickness $2L_1$ and $2L_2$, respectively, as indicated by Equation (4-18).

In a manner similar to that described above, the solution for a three-dimensional block may be expressed as a product of three infinite-plate solutions for plates having the thickness of the three sides of the block. Similarly, a solution for a cylinder of finite length could be

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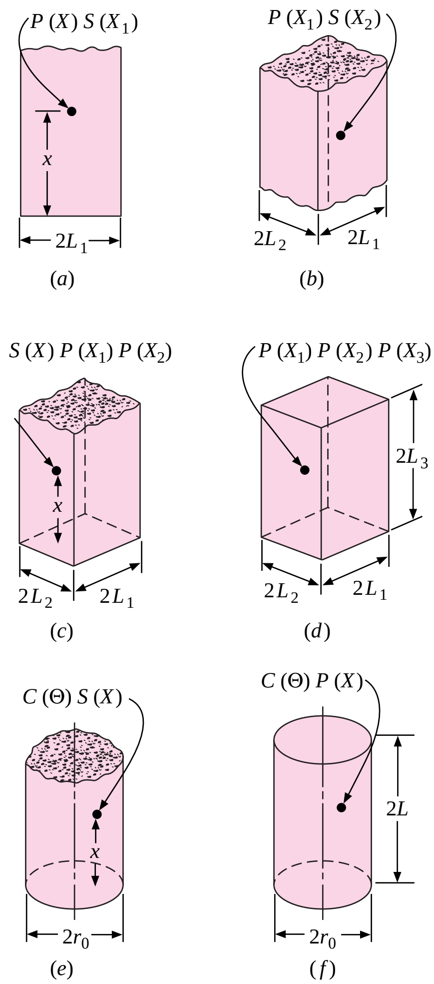
expressed as a product of solutions of the infinite cylinder and an infinite plate having a thickness equal to the length of the cylinder. Combinations could also be made with the infinite-cylinder and infinite-plate solutions to obtain temperature distributions in semi-infinite bars and cylinders. Some of the combinations are summarized in Figure 4-18, where

$$C(\Theta) = \text{solution for infinite cylinder}$$
$$P(X) = \text{solution for infinite plate}$$
$$S(X) = \text{solution for semi-infinite solid}$$

Figure 4-18 | Product solutions for temperatures in multidimensional systems: (*a*) semi-infinite plate; (*b*) infinite rectangular bar; (*c*) semi-infinite rectangular bar; (*d*) rectangular parallelepiped; (*e*) semi-infinite cylinder; (*f*) short cylinder.



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The general idea is then

$$\left(\frac{\theta}{\theta_i}\right)_{\substack{\text{combined}\\ \text{solid}}} = \left(\frac{\theta}{\theta_i}\right)_{\substack{\text{intersection}\\ \text{solid 1}}} \left(\frac{\theta}{\theta_i}\right)_{\substack{\text{intersection}\\ \text{solid 2}}} \left(\frac{\theta}{\theta_i}\right)_{\substack{\text{intersection}\\ \text{solid 3}}}$$

Heat Transfer in Multidimensional Systems

Langston [16] has shown that it is possible to superimpose the heat-loss solutions for one-dimensional bodies, as shown in Figures 4-14, 4-15, and 4-16, to obtain the heat for a multidimensional body. The results of this analysis for intersection of two bodies is

$$\left(\frac{Q}{Q_0}\right)_{\text{total}} = \left(\frac{Q}{Q_0}\right)_1 + \left(\frac{Q}{Q_0}\right)_2 \left[1 - \left(\frac{Q}{Q_0}\right)_1\right] \quad \textbf{[4-22]}$$

where the subscripts refer to the two intersecting bodies. For a multidimensional body formed by intersection of three one-dimensional systems, the heat loss is given by

$$\left(\frac{Q}{Q_0}\right)_{\text{total}} = \left(\frac{Q}{Q_0}\right)_1 + \left(\frac{Q}{Q_0}\right)_2 \left[1 - \left(\frac{Q}{Q_0}\right)_1\right] + \left(\frac{Q}{Q_0}\right)_3 \left[1 - \left(\frac{Q}{Q_0}\right)_1\right]\left[1 - \left(\frac{Q}{Q_0}\right)_2\right] \quad \textbf{[4-23]}$$

If the heat loss is desired after a given time, the calculation is straightforward. On the other hand, if the *time to achieve a certain heat loss* is the desired quantity, a trial-and-error or iterative procedure must be employed. The following examples illustrate the use of the various charts for calculating temperatures and heat flows in multidimensional systems.

Semi-Infinite Cylinder Suddenly Exposed to Convection

EXAMPLE 4-8

A semi-infinite aluminum cylinder 5 cm in diameter is initially at a uniform temperature of 200°C. It is suddenly subjected to a convection boundary condition at 70°C with $h = 525\ \text{W/m}^2 \cdot {}^\circ\text{C}$. Calculate the temperatures at the axis and surface of the cylinder 10 cm from the end 1 min after exposure to the environment.

Solution

This problem requires a combination of solutions for the infinite cylinder and semi-infinite slab in accordance with Figure 4-18*e*. For the slab we have

$$x = 10\ \text{cm} \qquad \alpha = 8.4 \times 10^{-5}\ \text{m}^2/\text{s} \qquad k = 215\ \text{W/m} \cdot {}^\circ\text{C}$$

so that the parameters for use with Figure 4-5 are

$$\frac{h\sqrt{\alpha\tau}}{k} = \frac{(525)[(8.4 \times 10^{-5})(60)]^{1/2}}{215} = 0.173$$

$$\frac{x}{2\sqrt{\alpha\tau}} = \frac{0.1}{(2)[(8.4 \times 10^{-5})(60)]^{1/2}} = 0.704$$

From Figure 4-5

$$\left(\frac{\theta}{\theta_i}\right)_{\text{semi-infinite slab}} = 1 - 0.036 = 0.964 = S(X)$$

For the infinite cylinder we seek both the axis- and surface-temperature ratios. The parameters for use with Figure 4-8 are

$$r_0 = 2.5\ \text{cm} \qquad \frac{k}{hr_0} = 16.38 \qquad \frac{\alpha\tau}{r_0^2} = 8.064 \qquad \frac{\theta_0}{\theta_i} = 0.38$$

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This is the axis-temperature ratio. To find the surface-temperature ratio, we enter Figure 4-11, using

$$\frac{r}{r_0} = 1.0 \qquad \frac{\theta}{\theta_0} = 0.97$$

Thus

$$C(\Theta) = \left(\frac{\theta}{\theta_i}\right)_{\text{inf cyl}} = \begin{cases} 0.38 & \text{at } r = 0 \\ (0.38)(0.97) = 0.369 & \text{at } r = r_0 \end{cases}$$

Combining the solutions for the semi-infinite slab and infinite cylinder, we have

$$\begin{aligned}\left(\frac{\theta}{\theta_i}\right)_{\text{semi-infinite cylinder}} &= C(\Theta)S(X) \\ &= (0.38)(0.964) = 0.366 \quad \text{at } r = 0 \\ &= (0.369)(0.964) = 0.356 \quad \text{at } r = r_0\end{aligned}$$

The corresponding temperatures are

$$T = 70 + (0.366)(200 - 70) = 117.6 \quad \text{at } r = 0$$
$$T = 70 + (0.356)(200 - 70) = 116.3 \quad \text{at } r = r_0$$

EXAMPLE 4-9 Finite-Length Cylinder Suddenly Exposed to Convection

A short aluminum cylinder 5.0 cm in diameter and 10.0 cm long is initially at a uniform temperature of 200°C. It is suddenly subjected to a convection environment at 70°C, and $h = 525$ W/m$^2 \cdot$°C. Calculate the temperature at a radial position of 1.25 cm and a distance of 0.625 cm from one end of the cylinder 1 min after exposure to the environment.

■ Solution

To solve this problem we combine the solutions from the Heisler charts for an infinite cylinder and an infinite plate in accordance with the combination shown in Figure 4-18*f*. For the infinite-plate problem

$$L = 5 \text{ cm}$$

The x position is measured from the center of the plate so that

$$x = 5 - 0.625 = 4.375 \text{ cm} \qquad \frac{x}{L} = \frac{4.375}{5} = 0.875$$

For aluminum

$$\alpha = 8.4 \times 10^{-5} \text{ m}^2/\text{s} \qquad k = 215 \text{ W/m} \cdot \text{°C}$$

so

$$\frac{k}{hL} = \frac{215}{(525)(0.05)} = 8.19 \qquad \frac{\alpha\tau}{L^2} = \frac{(8.4 \times 10^{-5})(60)}{(0.05)^2} = 2.016$$

From Figures 4-7 and 4-10, respectively,

$$\frac{\theta_0}{\theta_i} = 0.75 \qquad \frac{\theta}{\theta_0} = 0.95$$

so that

$$\left(\frac{\theta}{\theta_i}\right)_{\text{plate}} = (0.75)(0.95) = 0.7125$$

For the cylinder $r_0 = 2.5$ cm

$$\frac{r}{r_0} = \frac{1.25}{2.5} = 0.5 \qquad \frac{k}{hr_0} = \frac{215}{(525)(0.025)} = 16.38$$

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$$\frac{\alpha\tau}{r_0^2} = \frac{(8.4 \times 10^{-5})(60)}{(0.025)^2} = 8.064$$

and from Figures 4-8 and 4-11, respectively,

$$\frac{\theta_0}{\theta_i} = 0.38 \qquad \frac{\theta}{\theta_0} = 0.98$$

so that

$$\left(\frac{\theta}{\theta_i}\right)_{\text{cyl}} = (0.38)(0.98) = 0.3724$$

Combining the solutions for the plate and cylinder gives

$$\left(\frac{\theta}{\theta_i}\right)_{\text{short cylinder}} = (0.7125)(0.3724) = 0.265$$

Thus

$$T = T_\infty + (0.265)(T_i - T_\infty) = 70 + (0.265)(200 - 70) = 104.5°\text{C}$$

Heat Loss for Finite-Length Cylinder

EXAMPLE 4-10

Calculate the heat loss for the short cylinder in Example 4-9.

■ Solution

We first calculate the dimensionless heat-loss ratio for the infinite plate and infinite cylinder that make up the multidimensional body. For the plate we have $L = 5$ cm $= 0.05$ m. Using the properties of aluminum from Example 4-9, we calculate

$$\frac{hL}{k} = \frac{(525)(0.05)}{215} = 0.122$$

$$\frac{h^2\alpha\tau}{k^2} = \frac{(525)^2(8.4 \times 10^{-5})(60)}{(215)^2} = 0.03$$

From Figure 4-14, for the plate, we read

$$\left(\frac{Q}{Q_0}\right)_p = 0.22$$

For the cylinder $r_0 = 2.5$ cm $= 0.025$ m, so we calculate

$$\frac{hr_0}{k} = \frac{(525)(0.025)}{215} = 0.061$$

and from Figure 4-15 we can read

$$\left(\frac{Q}{Q_0}\right)_c = 0.55$$

The two heat ratios may be inserted in Equation (4-22) to give

$$\left(\frac{Q}{Q_0}\right)_{\text{tot}} = 0.22 + (0.55)(1 - 0.22) = 0.649$$

The specific heat of aluminum is 0.896 kJ/kg · °C and the density is 2707 kg/m^3, so we calculate Q_0 as

$$Q_0 = \rho c V \theta_i = (2707)(0.896)\pi(0.025)^2(0.1)(200 - 70)$$
$$= 61.9 \text{ kJ}$$

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The actual heat loss in the 1-min time is thus

$$Q = (61.9\text{ kJ})(0.649) = 40.2\text{ kJ}$$

4-6 | TRANSIENT NUMERICAL METHOD

The charts described in Sections 4-4 and 4-5 are very useful for calculating temperatures in certain regular-shaped solids under transient heat-flow conditions. Unfortunately, many geometric shapes of practical interest do not fall into these categories; in addition, one is frequently faced with problems in which the boundary conditions vary with time. These transient boundary conditions as well as the geometric shape of the body can be such that a mathematical solution is not possible. In these cases, the problems are best handled by a numerical technique with computers. It is the setup for such calculations that we now describe. For ease in discussion we limit the analysis to two-dimensional systems. An extension to three dimensions can then be made very easily.

Consider a two-dimensional body divided into increments as shown in Figure 4-19. The subscript m denotes the x position, and the subscript n denotes the y position. Within the solid body the differential equation that governs the heat flow is

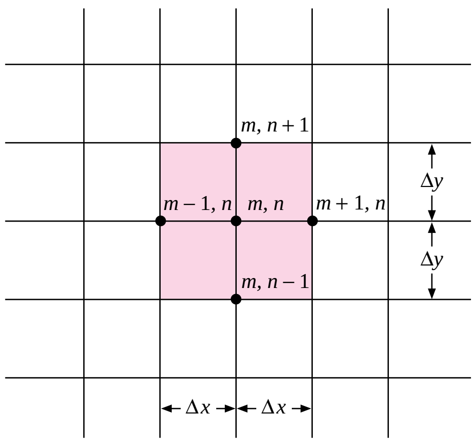
$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \rho c\frac{\partial T}{\partial \tau} \qquad \textbf{[4-24]}$$

assuming constant properties. We recall from Chapter 3 that the second partial derivatives may be approximated by

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{1}{(\Delta x)^2}(T_{m+1,n} + T_{m-1,n} - 2T_{m,n}) \qquad \textbf{[4-25]}$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{1}{(\Delta y)^2}(T_{m,n+1} + T_{m,n-1} - 2T_{m,n}) \qquad \textbf{[4-26]}$$

Figure 4-19 | Nomenclature for numerical solution of two-dimensional unsteady-state conduction problem.



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The time derivative in Equation (4-24) is approximated by

$$\frac{\partial T}{\partial \tau} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \quad \text{[4-27]}$$

In this relation the superscripts designate the time increment. Combining the relations above gives the difference equation equivalent to Equation (4-24)

$$\frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{(\Delta x)^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{(\Delta y)^2} = \frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta \tau} \quad \text{[4-28]}$$

Thus, if the temperatures of the various nodes are known at any particular time, the temperatures after a time increment $\Delta\tau$ may be calculated by writing an equation like Equation (4-28) for each node and obtaining the values of $T_{m,n}^{p+1}$. The procedure may be repeated to obtain the distribution after any desired number of time increments. If the increments of space coordinates are chosen such that

$$\Delta x = \Delta y$$

the resulting equation for $T_{m,n}^{p+1}$ becomes

$$T_{m,n}^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + \left[1 - \frac{4\alpha\,\Delta\tau}{(\Delta x)^2}\right]T_{m,n}^{p} \quad \text{[4-29]}$$

If the time and distance increments are conveniently chosen so that

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = 4 \quad \text{[4-30]}$$

it is seen that the temperature of node (m, n) after a time increment is simply the arithmetic average of the four surrounding nodal temperatures at the beginning of the time increment.

When a one-dimensional system is involved, the equation becomes

$$T_{m}^{p+1} = \frac{\alpha\,\Delta\tau}{(\Delta x)^2}\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + \left[1 - \frac{2\alpha\,\Delta\tau}{(\Delta x)^2}\right]T_{m}^{p} \quad \text{[4-31]}$$

and if the time and distance increments are chosen so that

$$\frac{(\Delta x)^2}{\alpha\,\Delta\tau} = 2 \quad \text{[4-32]}$$

the temperature of node m after the time increment is given as the arithmetic average of the two adjacent nodal temperatures at the beginning of the time increment.

Some general remarks concerning the use of numerical methods for solution of transient conduction problems are in order at this point. We have already noted that the selection of the value of the parameter

$$M = \frac{(\Delta x)^2}{\alpha\,\Delta\tau}$$

governs the ease with which we may proceed to effect the numerical solution; the choice of a value of 4 for a two-dimensional system or a value of 2 for a one-dimensional system makes the calculation particularly easy.

Once the distance increments and the value of M are established, the time increment is fixed, and we may not alter it without changing the value of either Δx or M, or both. Clearly, the larger the values of Δx and $\Delta\tau$, the more rapidly our solution will proceed. On the other hand, the smaller the value of these increments in the independent variables, the more accuracy will be obtained. At first glance one might assume that small distance increments could be used for greater accuracy in combination with large time increments