HW/Tutorial # 1
WWWR Chapters 15-16
ID Chapters 1-2

- Tutorial #1
- WWWR #15.25, 15.26, 15.1, 15.2, 15.3, 16.1, 16.2.
- ID # 1.11, 1.13.
- To be discussed during the week 17 Jan. – 21 Jan. , 2011.
- By either volunteer or class list.

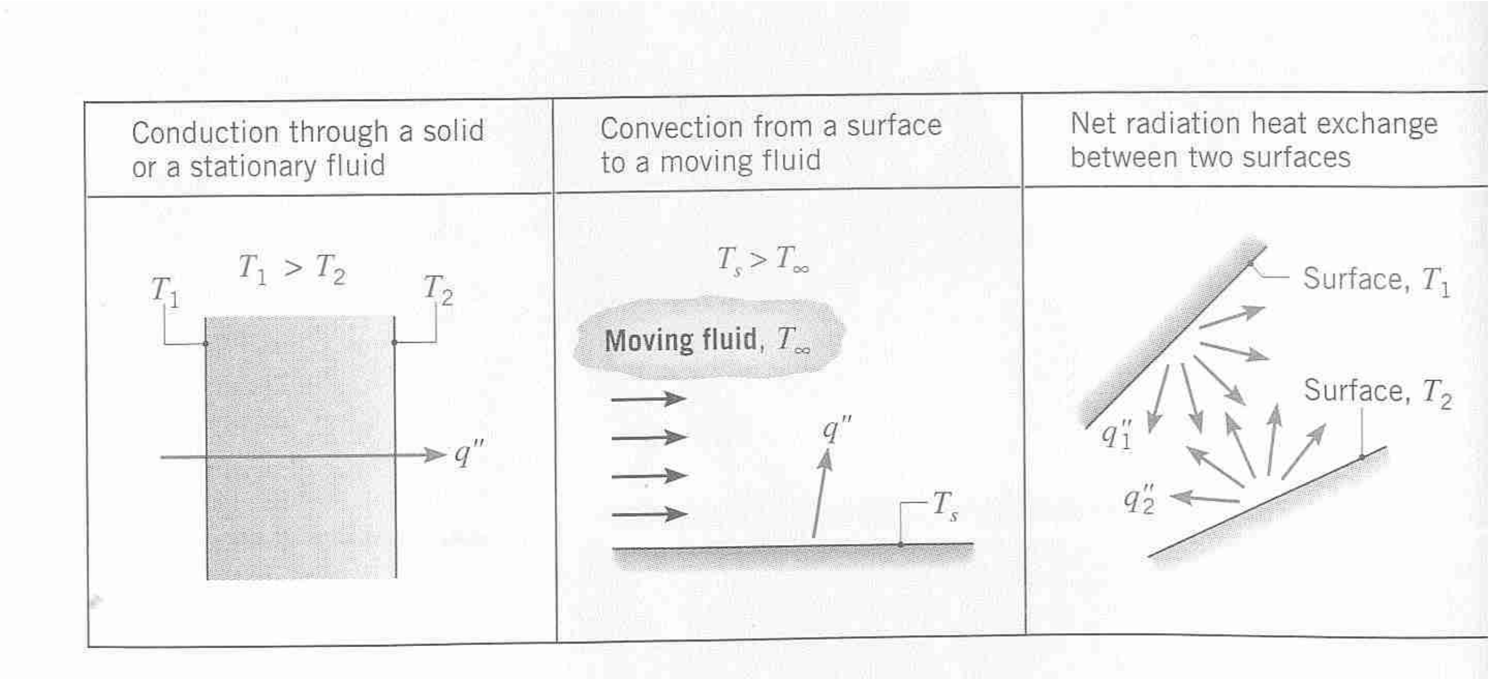
- HW # 1 (Self study – solution is provided)
- WWWR #15.15, 15.21, 15.22.
- ID # 2.2, 2.3.

Fundamentals of Heat Transfer

Conduction, Convection, and Radiation Heat Transfer Mode

Ref. ID Figure 1.1 (p 2)

Heat transfer (or heat) is thermal energy in transit due to a temperature difference



Conduction Heat Transfer

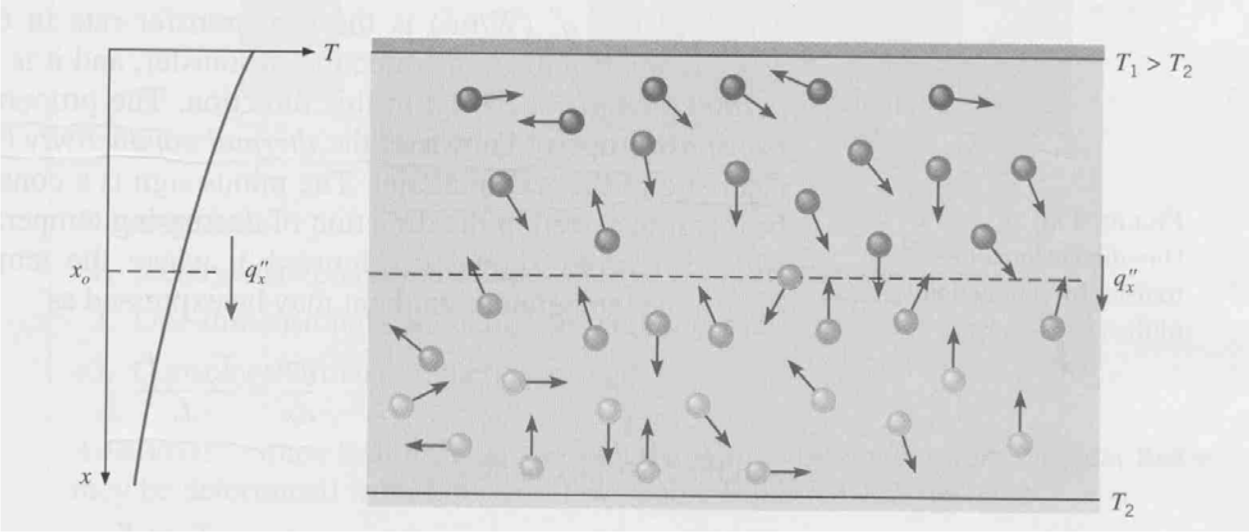
- First mechanism - molecular interaction (e.g. gas)
  - Greater motion of molecule at higher energy level (temperature) imparts energy to adjacent molecules at lower energy levels
- Second mechanism – by free electrons (e.g. solid)

$$\frac{q_x}{A} = -k\frac{dT}{dx}; \frac{q}{A} = -k\nabla T$$

Thermal Conductivity

- Physical origins and rate equation

- (Ref. ID; Figure 1.2) Association of conduction heat transfer with diffusion energy due to molecular activities.



Thermal Conductivity of Gas

- Estimation of the thermal conductivity of gas
- Ref. WWWR pp202-203 (Self Study)

- Derived from the gas kinetic theory:
- (1) Considering the summation of the energy flux associated with the molecules crossing the control surface;
- (2) The number of molecules involved is related to average random molecular velocity.
- (3) κ: Boltzmann constant, d: molecular diameter, m: mass per molecule.

$$k = \frac{1}{\pi^{1.5} d^2} \sqrt{\kappa^3 T / m}$$

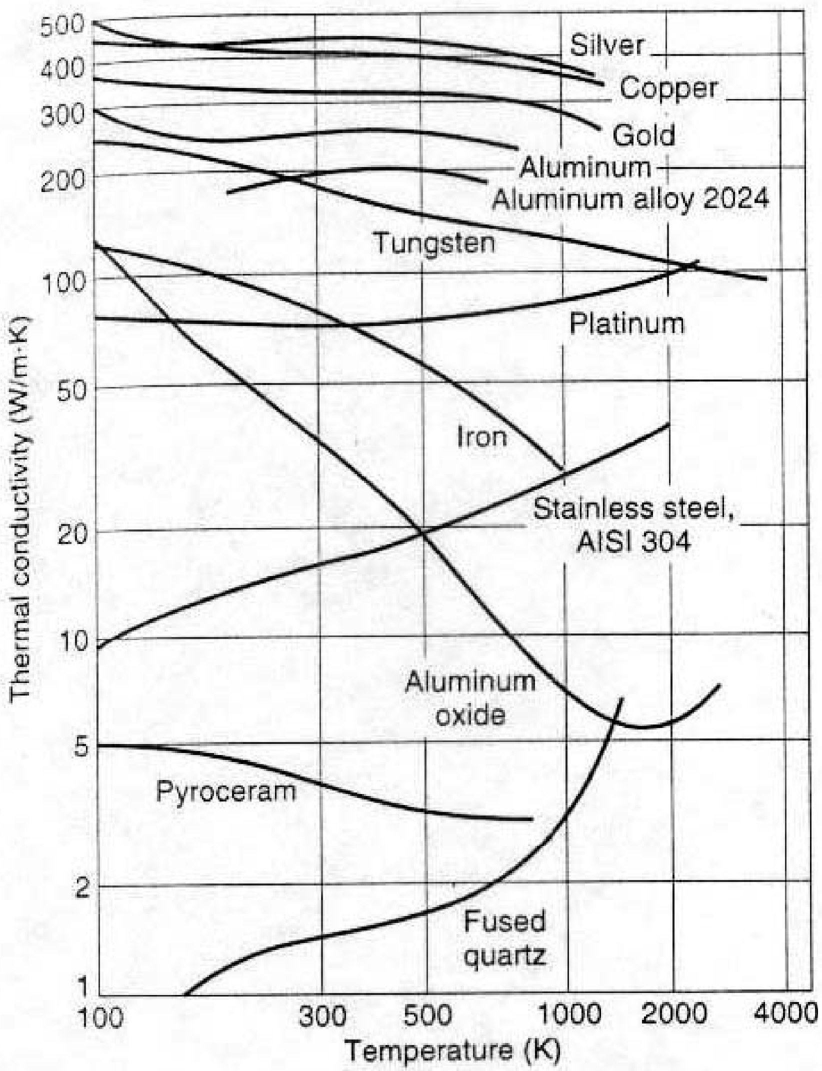
[Unit = W/(m-K)]

Thermal Conductivity of Solid

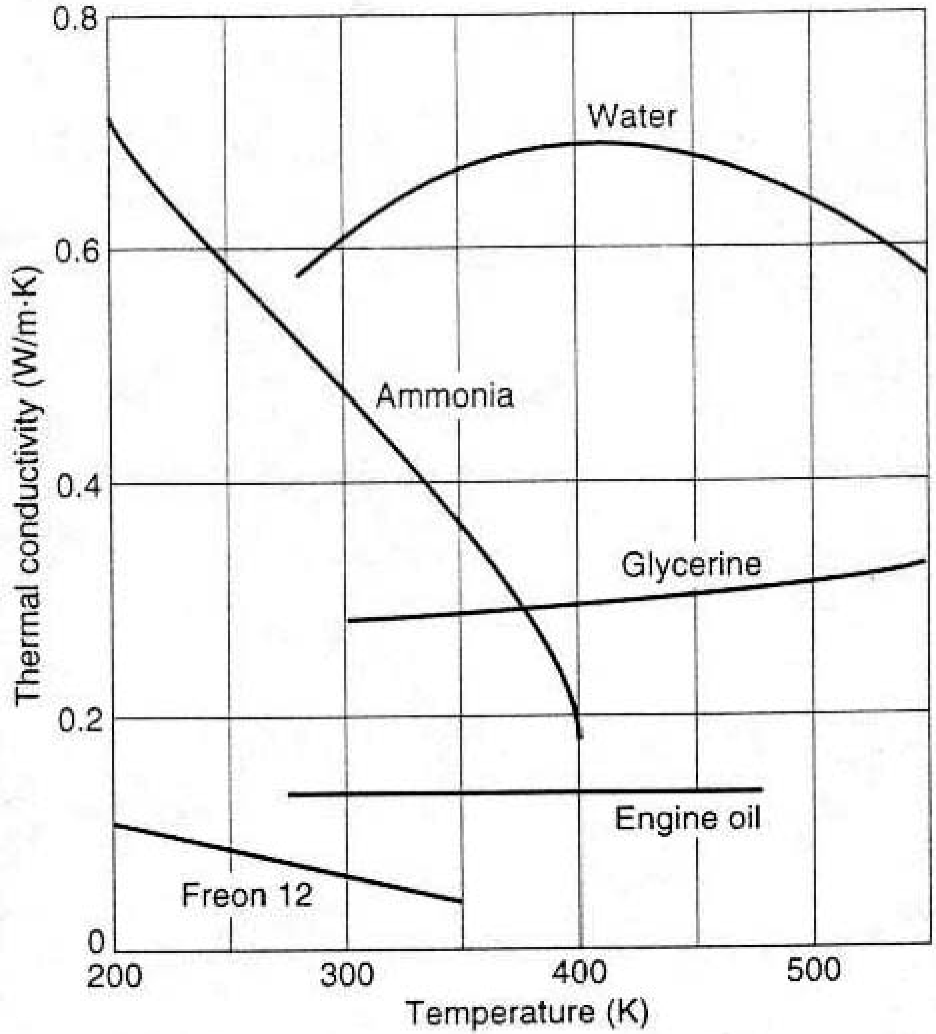
- Estimation of the thermal conductivity of solid
- Ref. WWWR pp204 (Self Study)

- (1) Derived from the Wiedemann, Franz, Lorenz Equation (1872).
- (2) The free electron mechanism of heat conduction is directly analogous to the mechanism of electric conduction.
- k_e : electrical conductivity [unit = 1/(Ω-m)], T: absolute temperature (unit = K), L: Lorenz number.

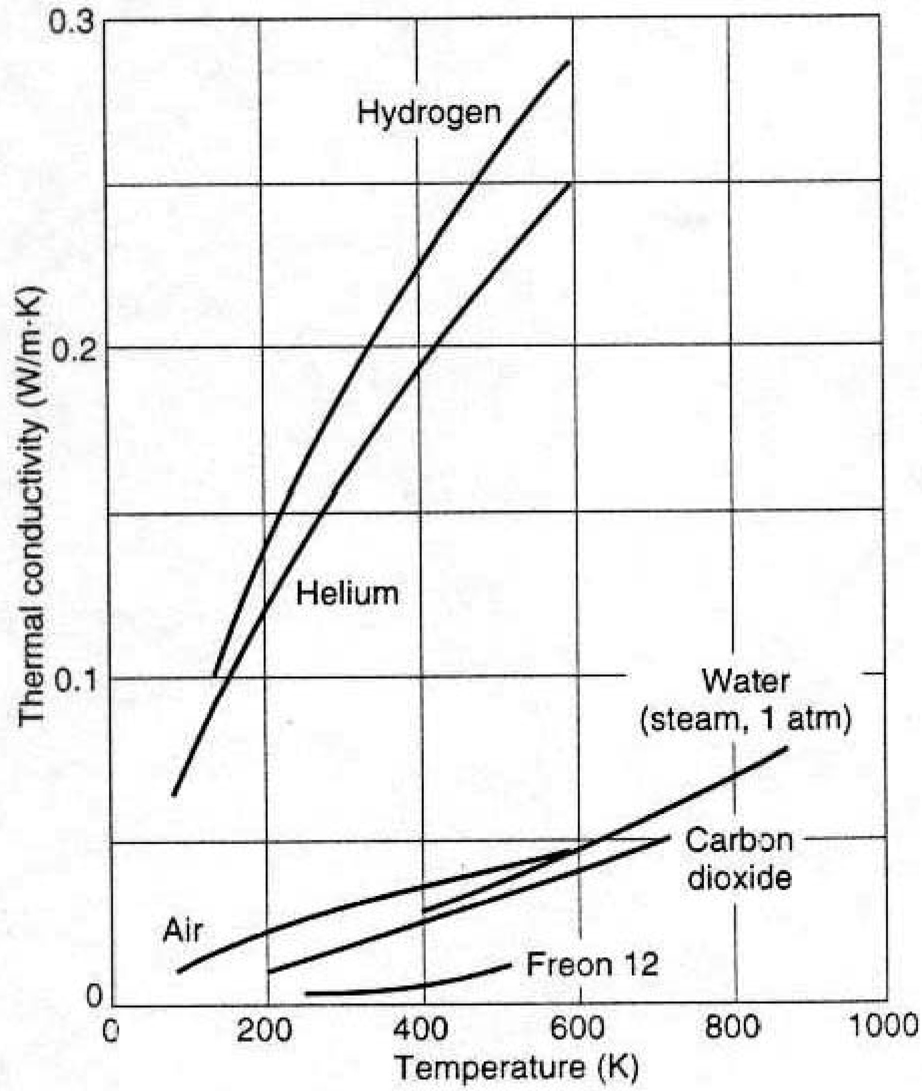
$$L = \frac{k}{k_e T} = \text{constant} \simeq 2.45 \cdot 10^{-8} W\Omega / K^2 @ 20^o C$$



(*a*) Solid materials



(*b*) Liquids



(*c*) Gases and vapors

Figure 15.2 Thermal conductivity of several materials at various temperatures.

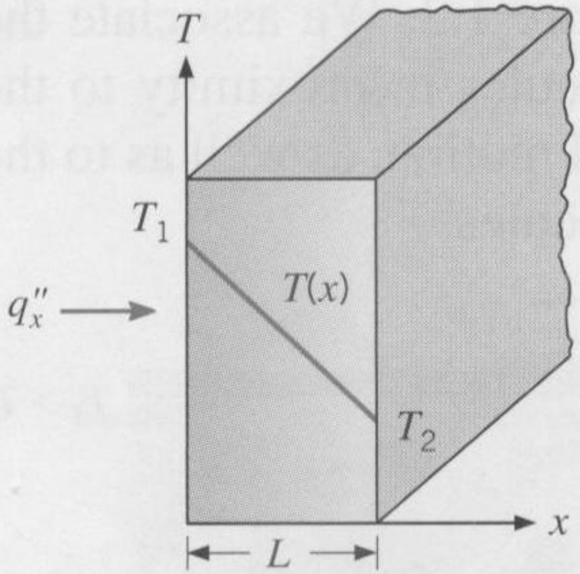


FIGURE 1.3
One-dimensional heat transfer by conduction (diffusion of energy).

$$q_x'' = -k\,\frac{dT}{dx}$$

The proportionality constant k is a transport property known as the thermal conductivity (W/mK) and is a characteristic of the wall material.

$$\frac{\mathrm{dT}}{\mathrm{dx}} = \frac{\mathrm{T}_2 - \mathrm{T}_1}{\mathrm{L}}$$

$$q_x'' = k\,\frac{T_1 - T_2}{L} = k\,\frac{\Delta T}{L}$$

EXAMPLE 1.1

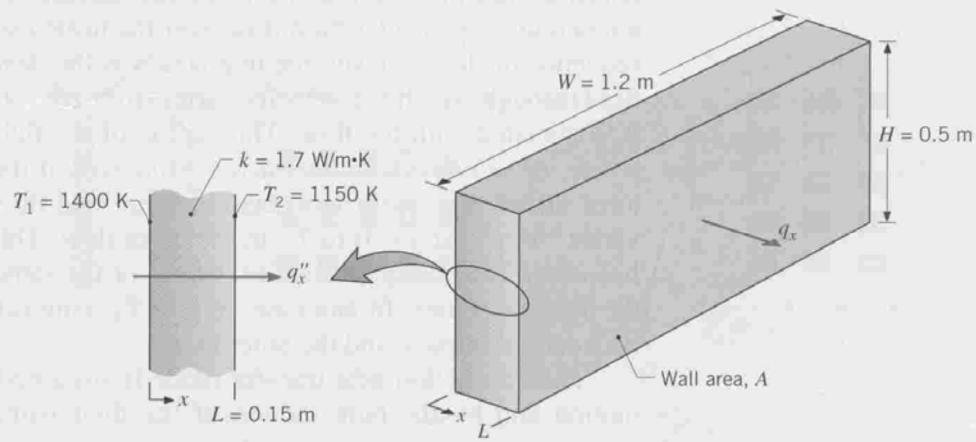
The wall of an industrial furnace is constructed from 0.15-m-thick fireclay brick having a thermal conductivity of 1.7 W/m · K. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall that is 0.5 m by 1.2 m on a side?

SOLUTION

Known: Steady-state conditions with prescribed wall thickness, area, thermal conductivity, and surface temperatures.

Find: Wall heat loss.

Schematic:



Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction through the wall.
3. Constant thermal conductivity.

Analysis: Since heat transfer through the wall is by conduction, the heat flux may be determined from Fourier's law. Using Equation 1.2, we have

$$q''_x = k\frac{\Delta T}{L} = 1.7\ \text{W/m}\cdot\text{K} \times \frac{250\ \text{K}}{0.15\ \text{m}} = 2833\ \text{W/m}^2$$

The heat flux represents the rate of heat transfer through a section of unit area, and it is uniform (invariant) across the surface of the wall. The heat loss through the wall of area $A = H \times W$ is then

$$q_x = (HW)\, q_x'' = (0.5\ \mathrm{m} \times 1.2\ \mathrm{m})\ 2833\ \mathrm{W/m^2} = 1700\ \mathrm{W} \qquad \triangleleft$$

Comments: Note the direction of heat flow and the distinction between heat flux and heat rate.

A steel pipe having an inside diameter of 1.88 cm and a wall thickness of 0.391 cm is subjected to inside and outside surface temperature of 367K and 344 K, respectively (see Figure 15.3). Find the heat flow rate per foot of pipe length, and also the heat flux based on both the inside and outside surface areas.

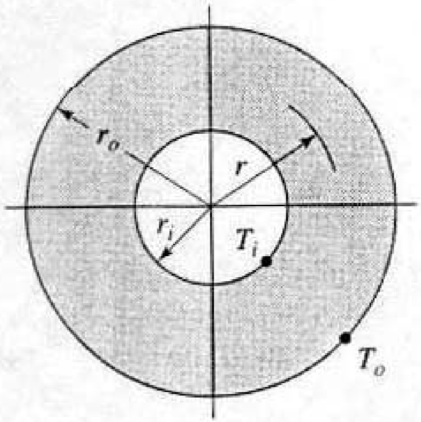


Figure 15.3 Heat conduction in a radial direction with uniform surface temperatures.

The first law of thermodynamics applied to this problem will reduce to the form $\delta Q/dt = 0$, indicating that the rate of heat transfer into the control volume is equal to the rate leaving i.e., $Q = q =$ constant.

Since the heat flow will be in the radial direction, the independent variable is r, and the proper form for the Fourier rate equation is

$$q_r = -kA\frac{dT}{dr}$$

Writing $A = 2\pi L r$, we see that the equation becomes

$$q_r = -k(2\pi rL)\frac{dT}{dr}$$

where q_r, is constant, which may be separated and solved as follows:

$$q_r \int_{r_i}^{r_o} \frac{dr}{r} = -2\pi kL \int_{T_i}^{T_o} dT = 2\pi kL \int_{T_o}^{T_i} dT$$

$$q_r \ln \frac{r_o}{r_i} = 2\pi kL\,(T_i - T_o)$$

$$q_r = \frac{2\pi kL}{\ln\ r_o/r_i}\,(T_i - T_o)$$

Substituting the given numerical values, we obtain

$$q_r = \frac{2\pi(42.90\ \text{W/m}\cdot\text{K})(367 - 344)\text{K}}{\ln(2.66/1.88)}$$

$$= 17\,860\ \text{W/m}\ (18\,600\ \text{Btu/hr}\cdot\text{ft})$$

The inside and outside surface areas per unit length of pipe are

$$A_i = \pi(1.88)(10^{-2})(1) = 0.059\ \text{m}^2/\text{m}\ (0.194\ \text{ft}^2/\text{ft})$$

$$A_o = \pi(2.662)(10^{-2})(1) = 0.084\ \text{m}^2/\text{m}\ (0.275\ \text{ft}^2/\text{ft})$$

Finally for the same amount of heat flow the fluxes based on
The inner and out surface areas differ by approximately 42%.

$$\frac{q_r}{A_i} = \frac{17860}{0.059} = 302.7 kW/m^2; \frac{q_r}{A_0} = \frac{17860}{0.084} = 212.6 kW/m^2$$

Consider a hollow cylindrical heat-transfer medium having inside and outside radii of r_i and r_o with the corresponding surface temperatures T_i and T_o. If the thermal-conductivity variation may be described as a linear function of temperature according to

$$k = k_0(1 + \beta T)$$

calculate the steady-state heat-transfer rate in the radial direction, using the above relation for the thermal conductivity, and compare the result with that using a k value calculated at the arithmetic mean temperature.

Figure 15.3 applies. The equation to be solved is now

$$q_r = -[k_o(1 + \beta T)](2\pi r L)\frac{dT}{dr}$$

which, upon separation and integration, becomes

$$q_r \int_{r_i}^{r_o} \frac{dr}{r} = -2\pi k_o L \int_{T_i}^{T_o} (1 + \beta T)\, dT$$

$$= 2\pi k_o L \int_{T_o}^{T_i} (1 + \beta T)\, dT$$

$$q_r = \frac{2\pi k_o L}{\ln r_o/r_i}\left[T + \frac{\beta T^2}{2}\right]_{T_o}^{T_i}$$

$$q_r = \frac{2\pi k_o L}{\ln r_o/r_i}\left[1 + \frac{\beta}{2}(T_i + T_o)\right](T_i - T_o) \qquad (15\text{-}10)$$

Noting that the arithmetic average value of k would be

$$k_{avg} = k_o\left[1 + \frac{\beta}{2}(T_i + T_o)\right]$$

we see that equation (15-10) could also be written as

$$q_r = \frac{2\pi k_{avg}L}{\ln r_o/r_i}(T_i - T_o)$$

Thus the two methods give identical results.

Convection:
Heat transfer due to convection involves the energy exchange between a surface and an adjacent fluid

Forced Convection: When a fluid is made to flow past a solid surface by an external agent such as a fan or pump

Free/Natural Convection: Warmer (or cooler) fluid next to the Solid boundary cause circulation because of the density variation Resulting from the temperature variation throughout a region of the fluid.

Newton's Law of Cooling: $q/A = h\Delta T$

q: rate of convective heat transfer (W); A: area normal to direction of

heat transfer; h: convective heat transfer coefficient, ΔT: temperature Difference between the surface and the fluid.

Convective Heat Transfer Processes: Ref: ID (Figure 1.5; p7)

(a) Forced Convection, (b) Free/Natural Convection, (c) Boiling, and (d) Condensation.

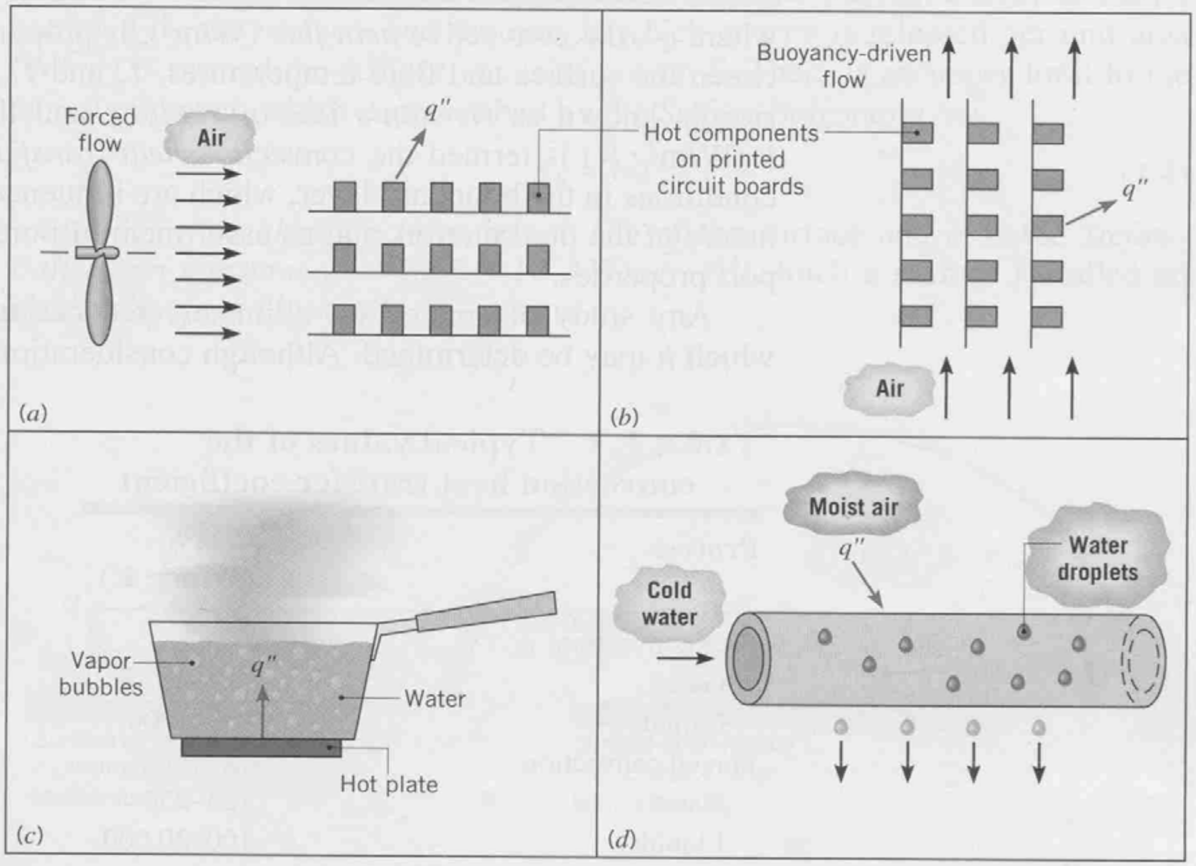
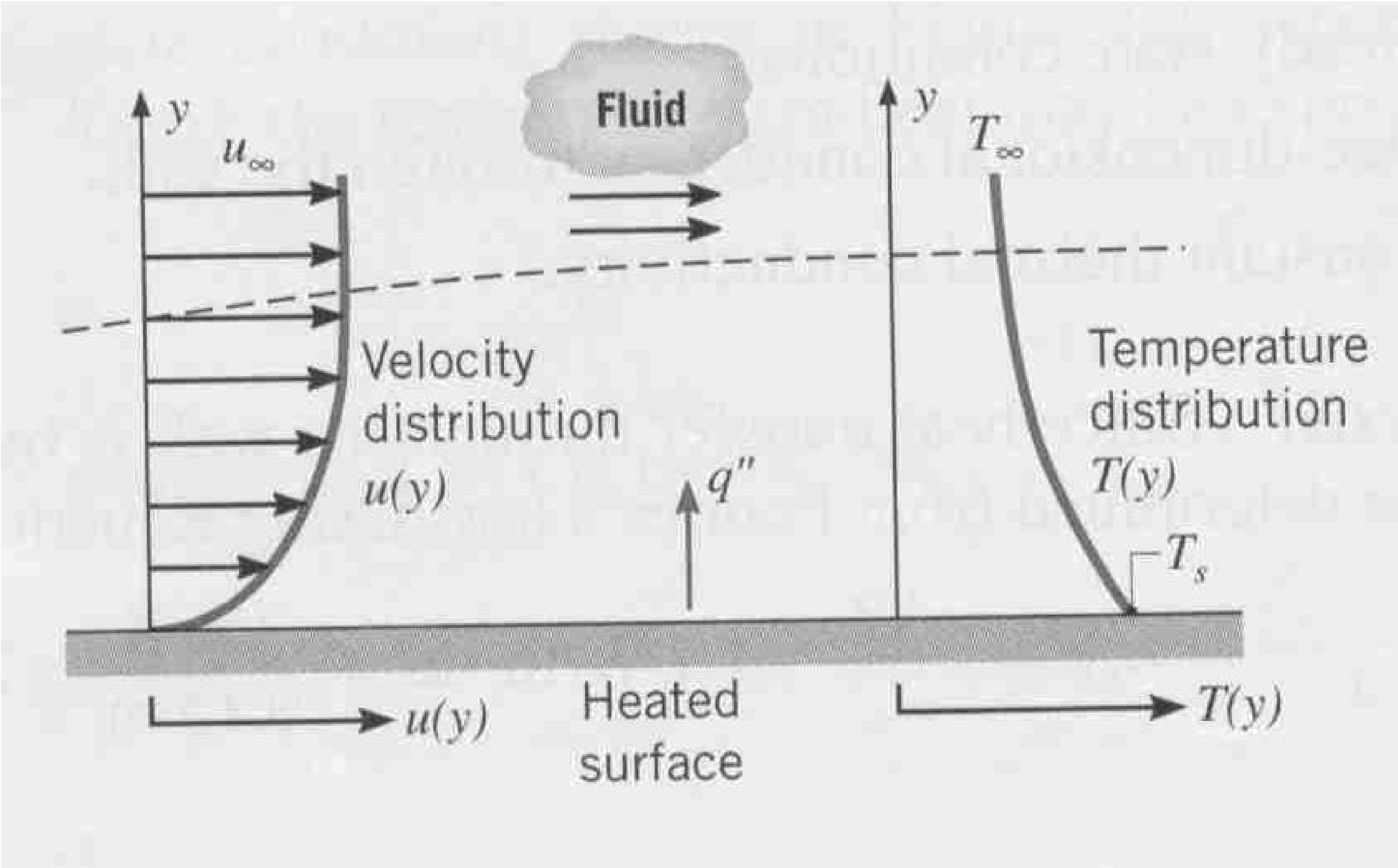


FIGURE 1.5 Convection heat transfer processes. (*a*) Forced convection. (*b*) Natural convection. (*c*) Boiling. (*d*) Condensation.

Boundary layer development in convection heat transfer

Ref. ID (P. 6; Fig. 1.4)



Radiant Heat Transfer

(1) No medium is required for its propagation.

(2) Energy transfer by radiation is maximum when the two Surfaces are separated by vacuum.

(3) Radiation heat transfer rate equation is given by the Stefan-Boltzmann law of thermal radiation:

$$\frac{q}{A} = \sigma T^4$$

q: rate of radiant energy emission (W); A: area of emitting surface (m^2); T: absolute temperature; σ: Stefan-Boltzmann Constant = 5.676×10^{-8} W/m^2-K^4

Radiation Exchange. Ref: ID (Figure 1.6; P. 9)

(a) At surface and (b) between a surface and large surroundings.

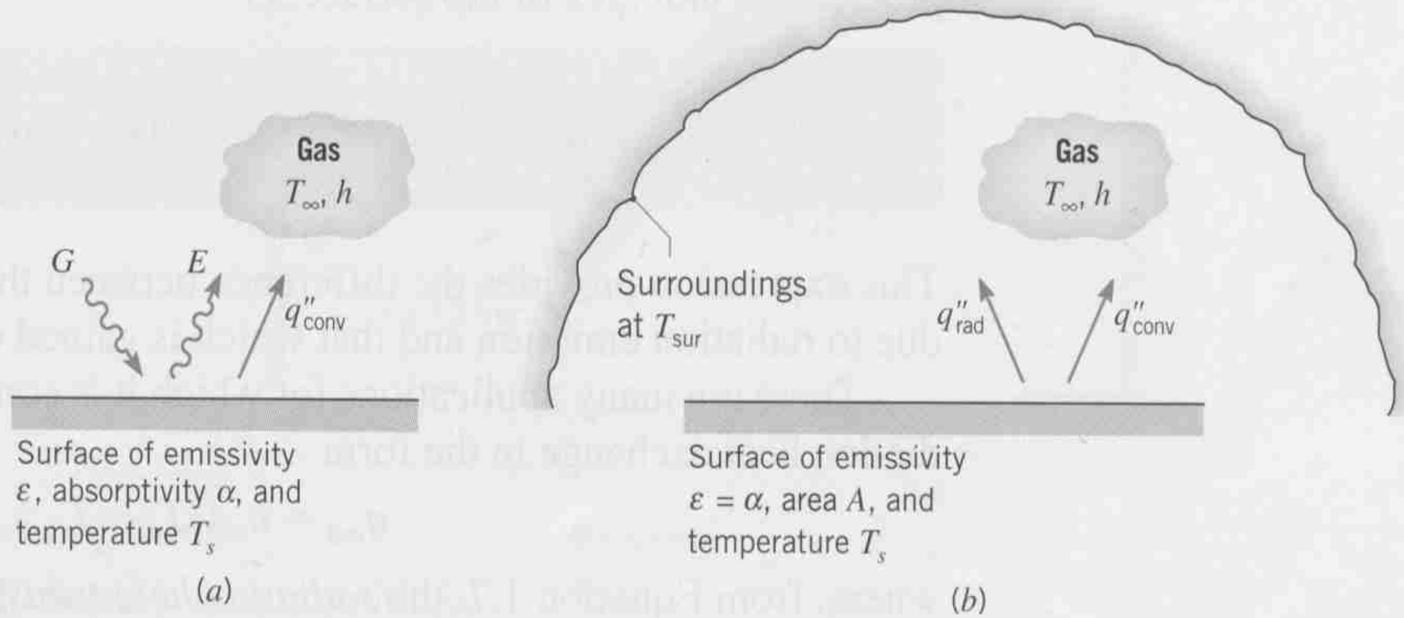


FIGURE 1.6 Radiation exchange: (*a*) at a surface and (*b*) between a surface and large surroundings.

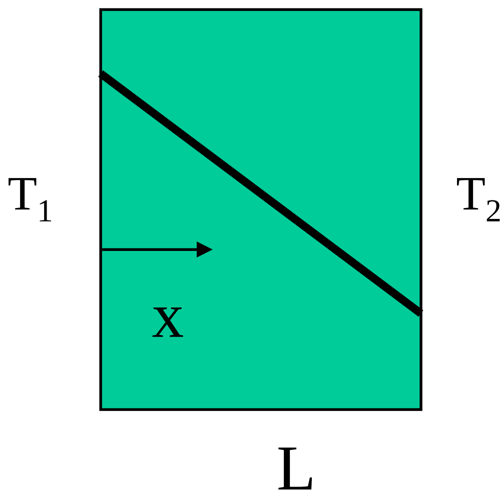
Table 15.1 Approximate Values of the Convective Heat-Transfer Coefficient

Mechanism	h, Btu/hr ft^2 °F	h, W/(m^2 · K)
Free convection, air	1–10	5–50
Forced convection, air	5–50	25–250
Forced convection, water	50–3000	250–15 000
Boiling water	500–5000	2500–25 000
Condensing water vapor	1000–20 000	5000–100 000

The three modes of heat transfer have been considered separated.

In real world, different modes of heat transfer are coupled.

Consider the case below for steady state conduction through a plane wall with its surfaces held at constant temperature T_1 and T_2.



Writing the Fourier rate equation for the x direction, we have

$$\frac{q_x}{A} = -k \frac{dT}{dx} \tag{15-1}$$

Solving this equation for q_x subject to the boundary conditions $T = T_1$ at $x = 0$ and $T = T_2$ at $x = L$ we obtain

$$\frac{q_x}{A} \int_0^L dx = -k \int_{T_1}^{T_2} dT = k \int_{T_2}^{T_1} dT$$

or

$$q_x = \frac{kA}{L}(T_1 - T_2) \tag{15-14}$$

Equation (15-14) bears an obvious resemblance to the Newton rate equation

$$q_x = hA\,\Delta T \tag{15-11}$$

We may utilize this similarity in form in a problem in which both types of energy transfer are involved.

Consider the composite plane wall constructed of three materials in layers with dimensions as shown in Figure 15.5. We wish to express the steady-state heat-transfer rate per unit area between a hot gas at temperature T_h on one side of this wall and a cool gas at T_c on the other side. Temperature designations and dimensions are as shown in the figure. The following relations for q_x arise from the application of equations (15-11) and (15-14):

$$q_x = h_h A(T_h - T_1) = \frac{k_1 A}{L_1}(T_1 - T_2) = \frac{k_2 A}{L_2}(T_2 - T_3)$$
$$= \frac{k_3 A}{L_3}(T_3 - T_4) = h_c A(T_4 - T_c)$$

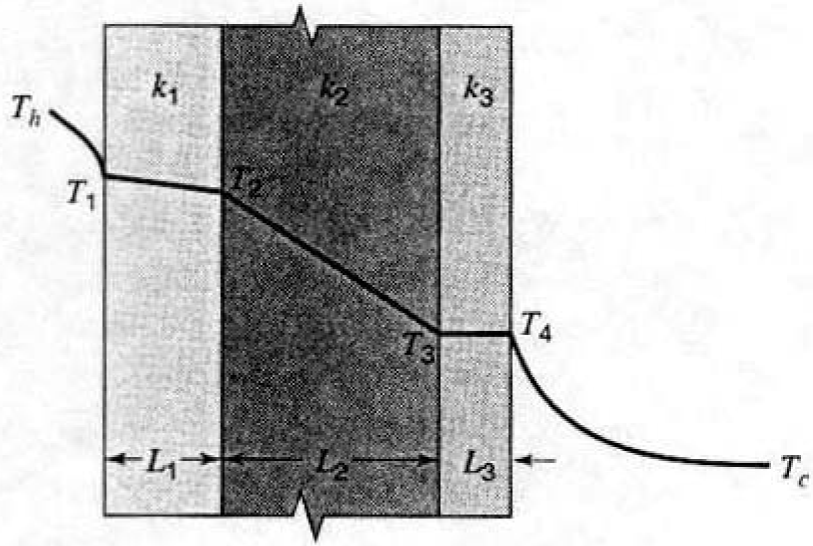


Figure 15.5 Steady-state heat transfer through a composite wall.

Each temperature difference is expressed in terms of q_x as follows:

$$T_h - T_1 = q_x(1/h_h A)$$
$$T_1 - T_2 = q_x(L_1/k_1 A)$$
$$T_2 - T_3 = q_x(L_2/k_2 A)$$
$$T_3 - T_4 = q_x(L_3/k_3 A)$$
$$T_4 - T_c = q_x(1/h_c A)$$

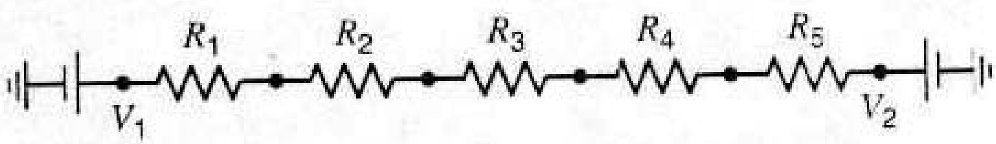
Adding these equations, we obtain

$$T_h - T_c = q_x\left(\frac{1}{h_h A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{L_3}{k_3 A} + \frac{1}{h_c A}\right)$$

and finally, solving for q_x, we have

$$q_x = \frac{T_h - T_c}{1/h_h A + L_1/k_1 A + L_2/k_2 A + L_3/k_3 A + 1/h_c A} \tag{15-15}$$

Note that the heat-transfer rate is expressed in terms of the *overall* temperature difference. If a series electrical circuit



is considered, we may write

$$I = \frac{\Delta V}{R_1 + R_2 + R_3 + R_4 + R_5} = \frac{\Delta V}{\sum R_i}$$

The analogous quantities in the expressions for heat flow and electrical current are apparent

$$\Delta V \to \Delta T$$
$$I \to q_x$$
$$R_i \to 1/hA,\ L/kA$$

and each term in the denominator of equation (15-15) may be thought of as a thermal resistance due to convection or conduction. Equation (15-15) thus becomes a heat-transfer analog to Ohm's law, relating heat flow to the overall temperature difference divided by the total thermal resistance between the points of known temperature. Equation (15-15) may now be written simply as

$$q = \frac{\Delta T}{\sum R_{\text{thermal}}} \tag{15-16}$$

This relation applies to steady-state heat transfer in systems of other geometries as well. The thermal-resistance terms will change in form for cylindrical or spherical systems, but once evaluated, they can be utilized in the form indicated by equation (15-16). With specific reference to equation (15-9), it may be noted that the thermal resistance of a cylindrical conductor is

$$\frac{\ln(r_o/r_i)}{2\pi kL}$$

Another common way of expressing the heat-transfer rate for a situation involving a composite material or combination of mechanisms is with the *overall heat-transfer coefficient* defined as

$$U \equiv \frac{q_x}{A\,\Delta T} \tag{15-17}$$

where U is the overall heat-transfer coefficient having the same units as h, in $W/m^2 \cdot K$ or $Btu/hr\ ft^2\ °F$.

Saturated steam at 0.276 MPa flows inside a steel pipe having an inside diameter of 2.09 cm and an outside diameter of 2.67 cm. The convective coefficients on the inner and outer pipe surfaces may be taken as 5680 $W/m^2 \cdot K$ and 22.7 $W/m^2 \cdot K$, respectively. The surrounding air is at 294 K. Find the heat loss per meter of bare pipe and for a pipe having a 3.8 cm thickness of 85% magnesia insulation on its outer surface.

In the case of the bare pipe there are three thermal resistances to evaluate

$$R_1 = R_{\text{convection inside}} = 1/h_i A_i$$
$$R_2 = R_{\text{convection outside}} = 1/h_0 A_0$$
$$R_3 = R_{\text{conduction}} = \ln(r_0/r_i)/2\pi kL$$

For conditions of this problem these resistances have the values

$$R_1 = 1/[(5680 \text{ W/m}^2 \cdot \text{K})(\pi)(0.0209 \text{ m})(1 \text{ m})]$$
$$= 0.00268 \text{ K/W}$$
$$R_2 = 1/[(22.7 \text{ W/m}^2 \cdot \text{K})(\pi)(0.0267 \text{ m})(1 \text{ m})]$$
$$= 0.525 \text{ K/W}$$

and

$$R_3 = \frac{\ln(2.67/2.09)}{2\pi(42.9 \text{ W/m} \cdot \text{K})(1 \text{ m})}$$
$$= 0.00091 \text{ K/W}$$

The inside temperature is that of 0.276 MPa saturated steam, 267°F or 404 K. The heat transfer rate per meter of pipe may now be calculated as

$$q = \frac{\Delta T}{\sum R} = \frac{404 - 294\ \text{K}}{0.528\ \text{K/W}}$$

$$= 208\ \text{W}$$

In the case of an insulated pipe, the total thermal resistance would include R_1 and R_3 evaluated above, plus additional resistances to account for the insulation. For the insulation

10.27 = 2.67+3.8*2

$$R_4 = \frac{\ln(10.27/2.67)}{2\pi(0.0675\ \text{W/m}\cdot\text{K})(1\ \text{m})}$$

$$= 3.176\ \text{K/W} \qquad \left(1.675\ \frac{\text{hr}\ {}^\circ\text{R}}{\text{Btu}}\right)$$

k value for 85% Magnesia WWWR Page 676 With interpolation

and for the outside surface of the insulation

$$R_5 = 1/[(22.7\ \text{W/m}^2\cdot\text{K})(\pi)(0.1027\ \text{m})(1\ \text{m})]$$

$$= 0.1365\ \text{K/W} \qquad \left(0.0720\ \frac{\text{hr}\ {}^\circ\text{R}}{\text{Btu}}\right)$$

thus the heat loss for the insulated pipe becomes

$$q = \frac{\Delta T}{\Sigma R} = \frac{404 - 294\ \text{K}}{3.316\ \text{K/W}}$$

$$= 33.2\ \text{W} \qquad \left(113\ \frac{\text{Btu}}{\text{hr}}\right)$$

a reduction of approximately 85%!

Example 3 could also have been worked by using an overall heat-transfer coefficient, which would be, in general

$$U = \frac{q_x}{A\,\Delta T} = \frac{\Delta T/\Sigma R}{A\,\Delta T} = \frac{1}{A \sum R}$$

or, for the specific case considered

$$U = \frac{1}{A\{1/A_i h_i + [\ln(r_o/r_i)]/2\pi kL + 1/A_o h_o\}} \tag{15-18}$$

Equation (15-18) indicates that the overall heat-transfer coefficient, U, may have a different numerical value, depending on which area it is based upon. If, for instance, U is based upon the outside surface area of the pipe, A_o, we have

$$U_o = \frac{1}{A_o/A_i h_i + [A_o \ln(r_o/r_i)]/2\pi kL + 1/h_o}$$

thus it is necessary, when specifying an overall coefficient, to relate it to a specific area.

One other means of evaluating heat-transfer rates is by means of the *shape factor*, symbolized S. Considering the steady-state relations developed for plane and cylindrical shapes

$$q = \frac{kA}{L}\,\Delta T \tag{15-14}$$

and

$$q = \frac{2\pi kL}{\ln(r_o/r_i)} \Delta T \tag{15-9}$$

if that part of each expression having to do with the geometry is separated from the remaining terms, we have, for a plane wall,

$$q = k\left(\frac{A}{L}\right) \Delta T$$

and for a cylinder

$$q = k\left(\frac{2\pi L}{\ln(r_o/r_i)}\right) \Delta T$$

Each of the bracketed terms is the shape factor for the applicable geometry. A general relation utilizing this form is

$$q = kS\,\Delta T \tag{15-19}$$

Equation (15-19) offers some advantages when a given geometry is required because of space and configuration limitations. If this is the case, then the shape factor may be calculated and q determined for various materials displaying a range of values of k.

The rate equations for heat transfer are as follows:

Conduction: the Fourier rate equation

$$\frac{\mathbf{q}}{A} = -k\,\nabla T$$

Convection: the Newton rate equation

$$\frac{q_x}{A} = h\,\Delta T$$

Radiation: the Stefan-Boltzmann Law for energy emitted from a black surface

$$\frac{q}{A} = \sigma T^4$$

Combined modes of heat transfer were considered, specifically with respect to the means of calculating heat-transfer rates when several transfer modes were involved. The three ways of calculating steady-state heat-transfer rates are represented by the equations

$$q_x = \frac{\Delta T}{\sum R_T} \tag{15-16}$$

where $\sum R_T$ is the total thermal resistance along the transfer path

$$q_x = UA\,\Delta T \tag{15-17}$$

where U is the overall heat transfer coefficient; and

$$q_x = kS\,\Delta T \tag{15-19}$$

where S is the shape factor.

The heat diffusion equation

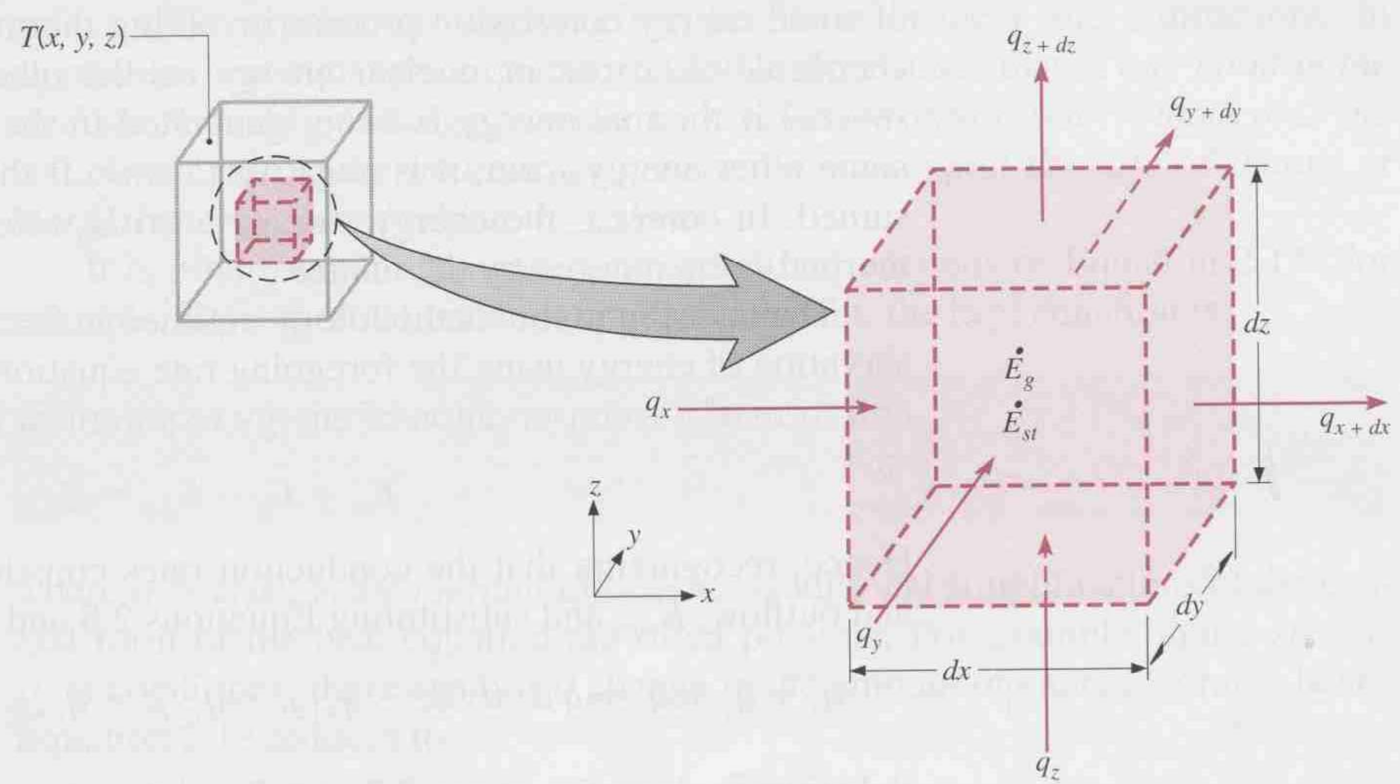


FIGURE 2.8 Differential control volume, *dx dy dz*, for conduction analysis in Cartesian coordinates.

Ref. ID (P. 61; Fig. 2.8)

Ref. ID (P. 62)

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx \qquad (2.7a)$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy \qquad (2.7b)$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz \qquad (2.7c)$$

Thermal energy generation

$$\dot{E}_g = \dot{q}\, dxdydz \qquad (2.8)$$

Energy storage

$$\dot{E}_{st} = \rho C_p \frac{\partial T}{\partial t} dxdydz \qquad (2.9)$$

Conservation of energy

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st} \qquad (1.11a)$$

$$q_x + q_y + q_z + \dot{q}\,dxdydz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho C_p \frac{\partial T}{\partial t} dxdydz \qquad (2.10)$$

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{q}\,dxdydz = \rho C_p \frac{\partial T}{\partial t} dxdydz \qquad (2.11)$$

$$q_x = -k dydz \frac{\partial T}{\partial x} \qquad (2.12\,a)$$

$$q_y = -k dxdz \frac{\partial T}{\partial y} \qquad (2.12\,b)$$

$$q_z = -k dxdy \frac{\partial T}{\partial z} \qquad (2.12\,c)$$

Heat (Diffusion) Equation: at any point in the medium the rate of energy transfer by conduction in a unit volume plus the volumetric rate of thermal energy must equal to the rate of change of thermal energy stored within the volume.

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho C_P\frac{\partial T}{\partial t} \qquad (2.13)$$

Net conduction heat flux into the controlled volume

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx = q_x'' - q''_{x+dx} \qquad (2.14)$$

If the thermal conductivity is constant.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.15)$$

Where $\alpha = k/(\rho\ C_p)$ is the thermal diffusivity

Under steady-state condition, there can be no change in the amount of energy storage.

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0 \qquad (2.16)$$

If the heat transfer is one-dimensional and there is no energy generation, the above equation reduces to

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \qquad (2.17)$$

Under steady-state, one-dimensional conditions with no energy generation, the heat flux is a constant in the direction of transfer.

Cylindrical coordinates (1)

When the del operator ∇ is expressed in cylindrical coordinates, the general form of the heat flux vector , and hence the Fourier's Law, is

$$q'' = -k\nabla T = -k(\vec{i}\frac{\partial T}{\partial r} + \vec{j}\frac{1}{r}\frac{\partial T}{\partial \phi} + \vec{k}\frac{\partial T}{\partial Z})$$

$$q_r'' = -k\frac{\partial T}{\partial r}; q_\varphi'' = -\frac{k}{r}\frac{\partial T}{\partial \varphi}; q_z'' = -k\frac{\partial T}{\partial z}$$

Cylindrical coordinates (2)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \quad (2.20)$$

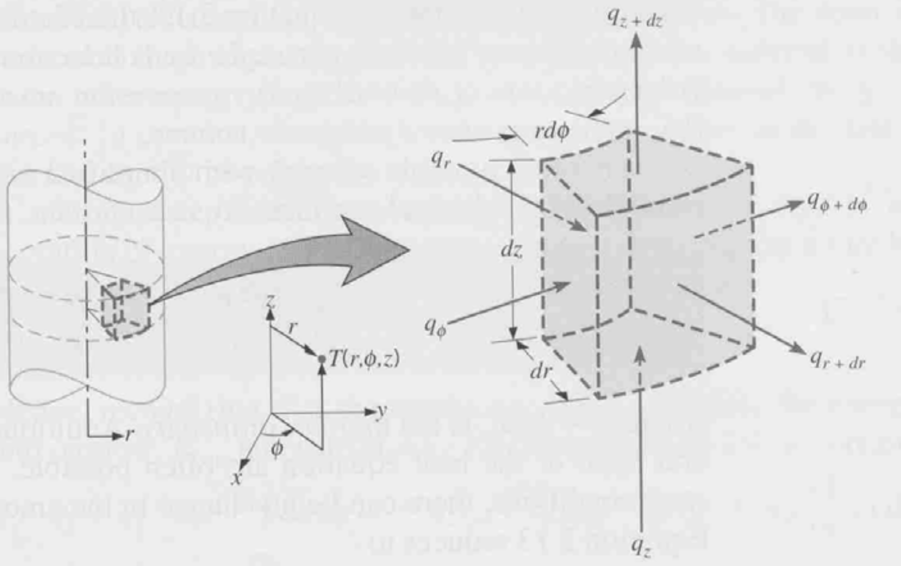


FIGURE 2.9 Differential control volume, $dr \cdot r\,d\phi \cdot dz$, for conduction analysis in cylindrical coordinates (r, ϕ, z).

Spherical coordinates (1)

When the del operator ∇ is expressed in spherical coordinates, the general form of the heat flux vector , and hence the Fourier's Law, is

$$q'' = -k\nabla T = -k(\vec{i}\,\frac{\partial T}{\partial r} + \vec{j}\,\frac{1}{r}\frac{\partial T}{\partial \theta} + \vec{k}\,\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi})$$

$$q_r'' = -k\frac{\partial T}{\partial r};\, q_\varphi'' = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \varphi};\, q_\theta'' = -\frac{k}{r}\frac{\partial T}{\partial \theta}$$

Spherical coordinates (2)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \quad (2.23)$$

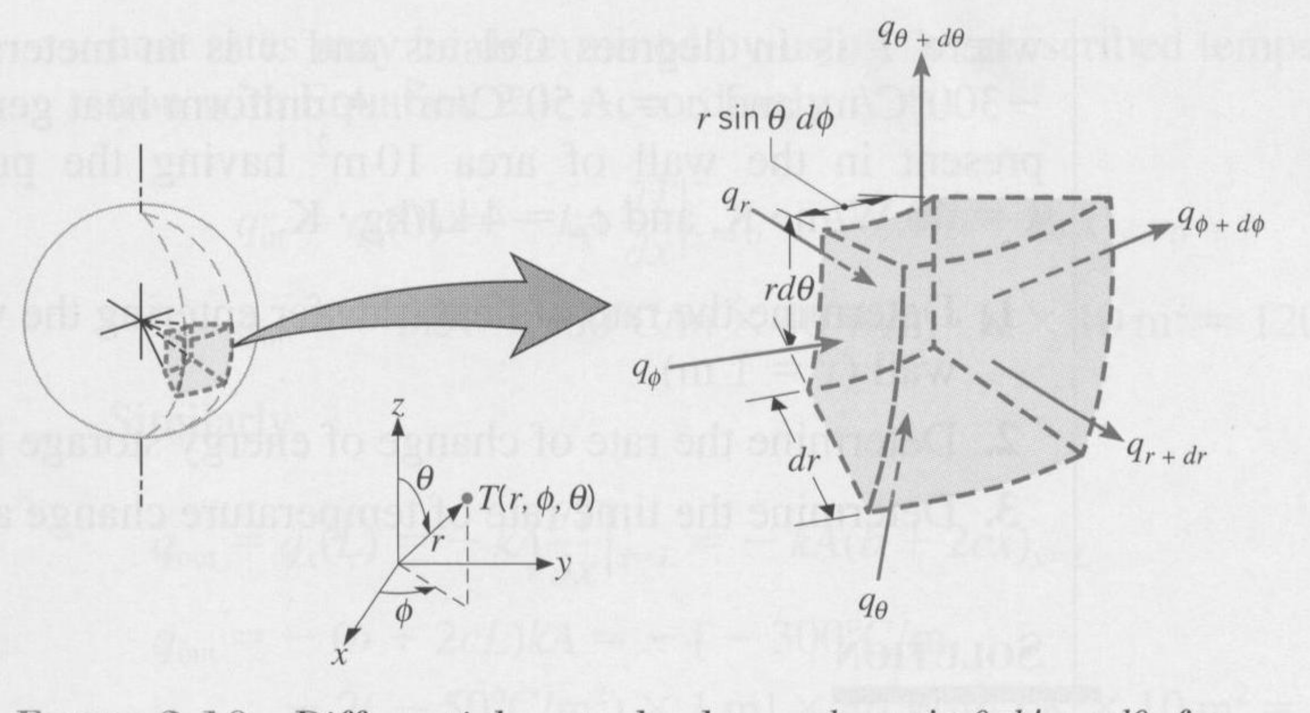


FIGURE 2.10 Differential control volume, $dr \cdot r\sin\theta\, d\phi \cdot r\, d\theta$, for conduction analysis in spherical coordinates (r, ϕ, θ).

Differential Equations of Heat Transfer

Consider the control volume having dimensions Δx, Δy, and Δz as depicted in Figure 16.1. Refer to the control-volume expression for the first law of thermodynamics

$$\frac{\delta Q}{dt} - \frac{\delta W_s}{dt} - \frac{\delta W_\mu}{dt} = \iint_{c.s.} \left(e + \frac{P}{\rho}\right) \rho (\mathbf{v} \cdot \mathbf{n})\, dA + \frac{\partial}{\partial t} \iiint_{c.v.} e\rho\, dV \qquad (6\text{-}10)$$

The individual terms are evaluated and their meanings are discussed below.

The net rate of heat added to the control volume will include all conduction effects, the net release of thermal energy within the control volume due to volumetric effects such as a chemical reaction or induction heating, and the dissipation of electrical or nuclear energy. The generation effects will be included in the single term, $\dot{q}$, which is the volumetric rate of thermal energy generation having units W/m^3 or Btu/hr ft^3. Thus the first term may be expressed as

$$\frac{\delta Q}{dt} = \left[k\frac{\partial T}{\partial x}\bigg|_{x+\Delta x} - k\frac{\partial T}{\partial x}\bigg|_{x} \right] \Delta y\, \Delta z + \left[k\frac{\partial T}{\partial y}\bigg|_{y+\Delta y} - k\frac{\partial T}{\partial y}\bigg|_{y} \right] \Delta x\, \Delta z$$

$$+ \left[k\frac{\partial T}{\partial z}\bigg|_{z+\Delta z} - k\frac{\partial T}{\partial z}\bigg|_{z} \right] \Delta x\, \Delta y + \dot{q}\, \Delta x\, \Delta y\, \Delta z \qquad (15\text{-}1)$$

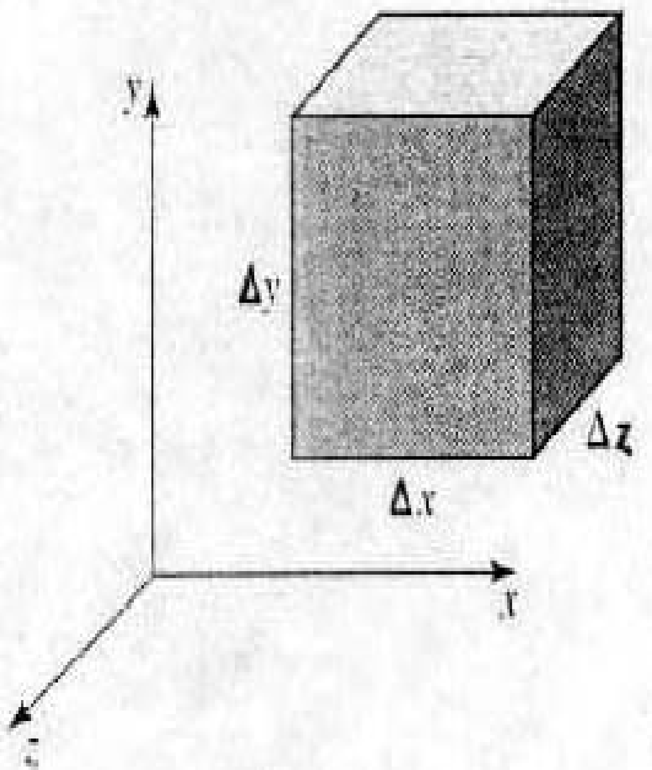


Figure 16.1 A differential control volume.

The shaft work-rate or power term will be taken as zero for our present purposes. This term is specifically related to work done by some effect within the control volume which, for the differential case, is not present. The power term is thus evaluated as

$$\frac{\delta W_s}{dt} = 0 \tag{16-2}$$

The viscous work rate, occurring at the control surface, is formally evaluated by integrating the dot product of the viscous stress and the velocity over the control surface. As this operation is tedious, we shall express the viscous work rate as $\Lambda\ \Delta x\ \Delta y\ \Delta z$ where Λ is the viscous work rate per unit volume. The third term in equation (6-10) is thus written as

$$\frac{\delta W_\mu}{dt} = \Lambda\ \Delta x\ \Delta y\ \Delta z \tag{16-3}$$

The surface integral includes all energy transfer across the control surface due to fluid flow. All terms associated with the surface integral have been defined previously. The surface integral is

$$\iint_{c.s.} \left(e + \frac{P}{\rho}\right) \rho(\mathbf{v} \cdot \mathbf{n})\, dA$$

$$\begin{aligned}
&= \left[\rho v_x\left(\frac{v^2}{2} + gy + u + \frac{P}{\rho}\right)\bigg|_{x+\Delta x} - \rho v_x\left(\frac{v^2}{2} + gy + u + \frac{P}{\rho}\right)\bigg|_{x}\right] \Delta y\ \Delta z \\
&+ \left[\rho v_y\left(\frac{v^2}{2} + gy + u + \frac{P}{\rho}\right)\bigg|_{y+\Delta y} - \rho v_y\left(\frac{v^2}{2} + gy + u + \frac{P}{\rho}\right)\bigg|_{y}\right] \Delta x\ \Delta z \\
&+ \left[\rho v_z\left(\frac{v^2}{2} + gy + u + \frac{P}{\rho}\right)\bigg|_{z+\Delta z} - \rho v_z\left(\frac{v^2}{2} + gy + u + \frac{P}{\rho}\right)\bigg|_{z}\right] \Delta x\ \Delta y
\end{aligned} \tag{16-4}$$

The energy accumulation term, relating the variation in total energy within the control volume as a function of time, is

$$\frac{\partial}{\partial t}\iiint_{\text{c.v.}} e\rho\, dV = \frac{\partial}{\partial t}\left[\frac{v^2}{2} + gy + u\right]\rho\, \Delta x\, \Delta y\, \Delta z \tag{16-5}$$

Equations (16-1) through (16-5) may now be combined as indicated by the general first-law expression, equation (6-10). Performing this combination and dividing through by the volume of the element, we have

$$\begin{aligned}
&\frac{k(\partial T/\partial x)|_{x+\Delta x} - k(\partial T/\partial x)|_x}{\Delta x} + \frac{k(\partial T/\partial y)|_{y+\Delta y} - k(\partial T/\partial y)|_y}{\Delta y} \\
&\qquad + \frac{k(\partial T/\partial z)|_{z+\Delta z} - k(\partial T/\partial z)|_z}{\Delta z} + \dot{q} + \Lambda \\
&= \frac{\{\rho v_x[(v^2/2) + gy + u + (P/\rho)]|_{x+\Delta x} - \rho v_x[(v^2/2) + gy + u + (P/\rho)]|_x\}}{\Delta x} \\
&\quad + \frac{\{\rho v_y[(v^2/2) + gy + u + (P/\rho)]|_{y+\Delta y} - \rho v_y[(v^2/2) + gy + u + (P/\rho)]|_y\}}{\Delta y} \\
&\quad + \frac{\{\rho v_z[(v^2/2) + gy + u + (P/\rho)]|_{z+\Delta z} - \rho v_z[(v^2/2) + gy + u + (P/\rho)]|_z\}}{\Delta z} \\
&\quad + \frac{\partial}{\partial t}\rho\left(\frac{v^2}{2} + gy + u\right)
\end{aligned}$$

General Form of The Differential Energy Equation

$$\nabla \bullet k \nabla T + \dot{q} + \phi = \rho C_v \frac{DT}{Dt}$$

conduction

dissipation

substantial derivative

transient + convective

Special Forms of The Differential Energy Equation

The applicable forms of the energy equation for some commonly encountered situations follow. In every case the dissipation term is considered negligibly small.

I. For an incompressible fluid without energy sources and with constant k

$$\rho c_v \frac{DT}{Dt} = k \nabla^2 T \tag{16-14}$$

II. For isobaric flow without energy sources and with constant k, the energy equation is

$$\rho c_v \frac{DT}{Dt} = k \nabla^2 T \tag{16-15}$$

Note that equations (16-14) and (16-15) are identical yet apply to completely different physical situations. The student may wish to satisfy himself at this point as to the reasons behind the unexpected result.

III. In a situation where there is no fluid motion all heat transfer is by conduction. If this situation exists, as it most certainly does in solids where $c_v \simeq c_p$, the energy equation becomes

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot k \nabla T + \dot{q} \tag{16-16}$$

Equation (16-16) applies in general to heat conduction. No assumption has been made concerning constant k. If the thermal conductivity is constant, the energy equation is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c_p} \tag{16-17}$$

where the ratio $k/\rho c_p$ has been symbolized by α and is designated the *thermal diffusivity*. It is easily seen that α has the units, L^2/t; in the SI system α is expressed in m^2/s, and as ft^2/hr in the English system.

If the conducting medium contains no heat sources, equation (16-17) reduces to the *Fourier field equation*

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{16-18}$$

which is occasionally referred to as Fourier's second law of heat conduction.

For a system in which heat sources are present but there is no time variation, equation (16-17) reduces to the *Poisson equation*

$$\nabla^2 T + \frac{\dot{q}}{k} = 0 \tag{16-19}$$

The final form of the heat-conduction equation to be presented applies to a steady-state situation without heat sources. For this case the temperature distribution must satisfy the *Laplace equation*

$$\nabla^2 T = 0 \tag{16-20}$$

Each of equations (16-17) through (16-20) has been written in general form, thus each applies to any orthogonal coordinate system. Writing the Laplacian operator, ∇^2, in the appropriate form will accomplish the transformation to the desired coordinate system. The Fourier field equation written in rectangular coordinates is

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] \tag{16-21}$$

in cylindrical coordinates

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] \tag{16-22}$$

and in spherical coordinates

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \right] \tag{16-23}$$

The reader is referred to Appendix B for an illustration of the variables in cylindrical and spherical coordinate systems.

Initial Conditions

Values of T and v at the start of time interval of interest.

Boundary Conditions

Values of T and v existing at specific positions of the boundaries of a system

i.e. for given values of the significant space variables

Isothermal Boundaries

Insulated Boundaries

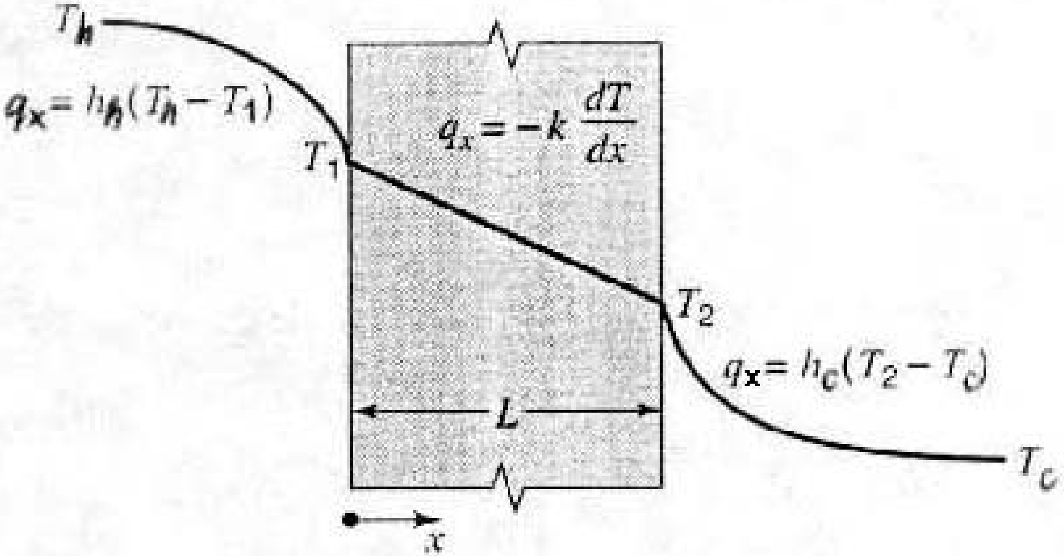


Figure 16.2 Conduction and convection at a system boundary.

This condition is illustrated in Figure 16.2. At the left-hand surface the boundary condition is

$$h_h(T_h - T|_{x=0}) = -k\left.\frac{\partial T}{\partial x}\right|_{x=0} \tag{16-24}$$

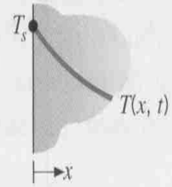
and at the right-hand surface

$$h_c(T|_{x=L} - T_c) = -k\left.\frac{\partial T}{\partial x}\right|_{x=L} \tag{16-25}$$

TABLE 2.1 Boundary conditions for the heat diffusion equation at the surface ($x = 0$)

1. Constant surface temperature

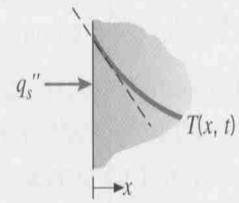
$$T(0, t) = T_s \qquad (2.24)$$



2. Constant surface heat flux

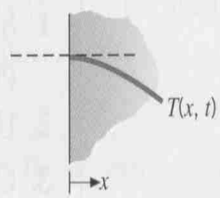
(a) Finite heat flux

$$-k \frac{\partial T}{\partial x}\bigg|_{x=0} = q_s'' \qquad (2.25)$$



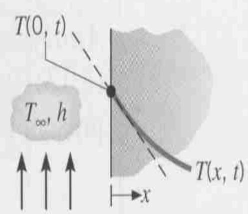
(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \qquad (2.26)$$



3. Convection surface condition

$$-k \frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)] \qquad (2.27)$$



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